# On the Versatility of a Math Major

## 多芸な数学系の学生になるために

**單維彰・2014-10-28**

### Introduction of the Speaker (Can be given by Yoshi)

Dear Morita san, colleagues of the Ryukoku University, dear students, and two of my superiors: Prof Lee and Prof Chern, thank you for being here and giving me the chance to express my opinions and share my thoughts on mathematics education on a college level. My name is 單維彰 (shan wei-zhang), and I work for the National Central University (國立中央大學) in Taiwan. I am an associate professor and adjunct professor appointed by the Department of Mathematics and Center of Teachers Training.

I have published two textbooks on math: one on Wavelets and the other on Calculus (in English), and there will be three books to be published in the near future: one on Fundamental Concepts of Computers, one on elementary and applied Calculus for business majors, and one on Mathematics in the Cultural Context.

In Teachers Training program, I have cultivated more than 50 math teachers for junior (中学校) and senior (高等学校) high schools. I have given lectures to more than 200 in-job math teachers each year. I have published one set of high school math textbooks in eight volumes. And I am now working with colleagues to design the high school math curriculum that goes into effect in 2018.

I am also serving the second term of the director of the Language Center in National Central University. The Center is responsible for the required courses and electives of English on campus. And it offers electives in other foreign languages including Japanese. We also have a Chinese program for international students as well as academic writing programs in English and Chinese for graduate students.

Finally, I am running the office of Educational Resource Center for the Counties of Taoyuan, Hsinchu, and Miaoli. It is one of the six Resource Centers financed directly by the Ministry of Education (教育部) in Taiwan. My office offers services to 12 universities and 48 high schools in the local area.

The reason that I become the director of the Language Center is probably because I always propose the following idea.

## Think of Mathematics as a Language

## 数学を言語として認識すれば

Everybody here speaks a native language, which is probably Japanese (日本語 Nihongo). Everybody has the experience of learning a foreign language, which is probably English. Mathematics is a language that describes forms, shapes, quantities and all kinds of relations among them.

As a matter of fact, mathematics is built into our natural language. Didn't each one of us start our math education with chanting 1, 2, 3, 4, to 100 (Hyaku) and beyond, without knowing the meaning of the numerals and number systems even before we can read or write? Then we use those numerals to count. And we build up our mathematical knowledge along with our daily language on How much is left? (余り分) How much comes short? (不足分) How much totally? (合わせる分) How much can be shared? (分ける分)　I bet no one in this room still remember how you learned these math. Because they came so natural to you and you did not feel any difficulty with them.

So please admit this: We are born with some mathematical abilities. They are learned so easily as if we had already known them and we went school only to recall them. That is why Westerners say "God creates natural numbers." When we cannot explain the mystery, we usually resort to gods (神).

But what I said is certainly not convincing at all. If we are born with the ability, then why on earth after some point, mathematics becomes so hard? It does not even look like a foreign language, instead it looks like an alien language (宇宙人語)? There are many ways to approach this pragmatic and interesting question, and there are many topics open for further research. I cannot dig into the details now, but have to content myself with proposing an analogy.

You all speak and write Japanese, right? How many of you can write Haiku (俳句)? I am sure you can read poems, but are you able to understand and to appreciate them? We all tell stories. We recount the dreams we had last night, we share the happiness and sorrows of our childhood; we express our wishes for the next Summer vacation. But how many of us can create a great story as 小川洋子(Ogawa Yo:ko) did in 博士

の愛した数式?

I hope I have made my point. There are so many different layers of language acquisition and mastery. Your professors sitting in the front row are like Haiku poets (俳人) in mathematics. They create new knowledge, new structure, and new ways of expression in mathematics. From the counting of 1, 2, 3, 4 to the level of your professors, there are all different layers.

As math majors you may become one of them in the near future. But many of you know that you are going to live other lives. You also know that, to be frank with you, it is virtually impossible for the Department to design a curriculum that suits the personal need for each one of you. It is only reasonable for each professor to present the best part of mathematics to you, and it is up to you to choose what to grasp and how much to grasp from the courses in order to prepare yourself as versatile as possible.

Let’s start with the preparation for being a math teacher for junior and senior high schools.

## Math Education: Practice and Philosophy

## 数学教育の実践と理念

The most notable characteristics for the learning of native languages (for instance, Japanese) are, first of all, we pick up the language in an immersion of a realistic environment, and secondly, we learn by examples. You don’t pick up Japanese by studying the definition of words and rules of grammar; instead you just pick up the language by listening to others and by trial-and-error. Besides, you acquire Japanese in an environment where you get the responses of the language practice almost immediately. And it is amazing that we do not need a lot of examples to conclude a syntactic rule in learning the native language.

The situation for learning most elementary mathematics is pretty much like picking up the native language. At the outset, mathematics is closer to our native language, and our instincts are working well. But gradually the mathematics environment becomes foreign to us and we started to learn it by strange rules without sensible examples. Mathematics becomes a foreign language. Maybe it is eventually inevitable, but I think we should postpone its happening for as long as possible.

This is the most fundamental idea of math education I want to share: Try best to keep it behaving like the native language. Learn in an immersion environment, and teach by examples.

Mathematics is a man-made universe. Here the word Man stands for the human being; it has nothing to do with genders. How can we immerse ourselves in this man-made world? We are lucky that now we have another man-made world that kids are fond of: Computer Games. I think the old-fashioned board games and card games are also man-made worlds. I am not saying that we take the trouble to create games for mathematics learning. Certainly there are people who devote themselves to this mission, but I think it is more of an attitude than of an action.

There are roles in a game. When you update a game, you must keep the game compatible with the old versions and you usually introduce new roles into the game. The new characters must interact with the old ones, so that they inherit most existing rules. But the new characters must have their own functions, so that they introduce new rules or bring about exceptions. This is just the case when junior high school students meet negative numbers (負数) for the first time.

The whole numbers, positive integers and zero, are like the old characters in an old game. Now we update the game to a new (exciting and hot) version, and here comes the new (brilliant and sexy) characters, negative integers. Together, they become the mighty Integer that is even faster and stronger.

While God creates positive integers, which means all others are our own creations. We create negative integers just as we create card games. So what are negative integers? We create them on the number line. Given that the students know the half number line with whole numbers, we create $-1$ to be the opposite number of 1 with respect to 0, and similarly $-2$ is opposite to 2, and so three and so forth. Let the kids develop their own examples.

Now that we created negative integers, how shall we play with them? We set the rules for negative numbers. But, like a game, we don't set a lot of rules in detail, it will spoil the fun. We set a small number of rules and let you enjoy the game. What are the rules we set for negative numbers? You say minus means the opposite. According to this rule, what is $-(-1)$ or what is the opposite of $-1$? And you set the rule that anything plus a negative number means counting along the number line to the left: the opposite direction of adding a positive number. For instance let's play $3+(-1)$, then

$5+(-3)$, then $4+(-6)$, and how about $(-2)+(-1)$?

Only a small number of examples are sufficient for the kids to start playing with negative integers on their own. Don't forget we are born with certain amount of mathematical instincts. There is a Chinese saying that goes like 三人成虎 which means, if you meet three persons consecutively and they all tell you that they saw a tiger in your backyard, you are going to believe it. That means, no matter how absurd an idea is, three examples are enough for you to believe it. People can usually derive a general rule by very few examples. And people can usually do something without knowing the reason, just like a young kid mimicking the adult language. Don't underestimate the power of teaching by examples, especially in the elementary and junior high school levels.

How can we keep the students in an immersion environment? The answer is resort to what they already know. Negative numbers inherit the old rule that you can switch the order of addition. Students usually do it intuitively, and you can leave them alone and don't bother to explain why. There are 8 cases when positive or negative numbers are adding or subtracting together. To help your kids win the game, you direct them to strategies (攻略). The best strategy is to fall back on what they already know. After some experiments and examples, you lead your squad to conclude that all of the 8 cases can be reduced to only two kinds of old styled arithmetic: $a+b$ where $a$ and $b$ are positive, and $a-b$ where $a$ and $b$ are positive and $a>b$.

When the students have mastered the new game, it's time for you to get serious. Now it's time to check them out: what negative numbers can do for us in the real world? There are so many models to show. I guess the key points are, first, whenever there is a direction in the measurement, with a reference center, it can be measured by positive and negative numbers. Take temperature and financing as examples. Secondly, negative numbers eliminate the necessity of subtraction. All subtraction can be replaced by adding the opposite number. And why do we care about that? Take financing for example. If your business has an asset last month or it got into a debt last month, and you earn something this month or you lose something this month. If you have only positive numbers, there will be four different procedures to compute the asset or the debt for this month. With positive and negative numbers, there is only one procedure. It is conceptually easy and it makes computer programs much more efficient.

Let me talk about language again. Some of your students may become a Nobel

Laureate in Literature (ノーベル文学賞), some may win a Fields Medal (フィールド賞). But you know the chance is slim. Most of your students come to your class to learn mathematics as a language: to read, to speak, and to reason with the common sense. Please keep this in mind and train yourself to do the job professionally.

## Math Teacher as a Profession　職業としての数学教師

What makes the profession as a math teacher? Let me point out that, first, to be an efficient test-problem solver does not make you a professional math teacher. Secondly, to love and care about your students do not count for a profession.

I believe you did well in school tests and you can solve high school test problems. Let's take the college entrance examination (大学入試) as an example. If solving those problems makes a math teacher, then anyone who achieve the full score in the test can be a math teacher. But do you agree that anyone that scores higher than you in the test can take over your job? Similarly, do you agree that anyone who loves the student more can replace you as a math teacher? Maybe his mother or maybe her boyfriend will do a better job.

It may sound cynical. I am not worrying about your knowledge and skills on doing high school math. What concerns me is that you know too much and do it too well. Being a professional math teacher, you need to know what not to do. Don't say it just because you know it. Don't do it simply because it is technically doable. Don't give counterexamples before the students are given plenty of examples. Don't prove when there is no doubt.

But I would like to add immediately that it does not include you. You are math majors, mathematics is your profession, you are obliged to prove.

A common saying has it that Timing is Everything . A professional math teacher knows not only what to say but more importantly when to say. To achieve this goal, you need not only to make yourself acquaint with the contents and test problems of high school math, which I am sure you already did, but also you need to know the context.

The word Context simply means what comes before and what goes after. The Rule Number One of education says that what makes a whole world of difference in the outcome of a lecture is what students already know before the lecture. Knowing the context of the curriculum gives you the ability to provide students with most essential reviews or re-enforcements. For instance, you know that what matters for learning negative integers is the prerequisite of the half-number line, and the proficiency of doing $a+b$ and $a-b$ where a and b are positive and $a \geq b$. So if your pupil has difficulties getting negative integers, you don't have to repeat everything in elementary school and you don't repeat the new materials again and again, you can fix his problem by going back to the source.

And high school math is not the purpose in itself. Topics in the curriculum always have their reason to be. Knowing the context gives you the ability to judge the purpose of the topic. And it guides you to select the materials that best suit your students. For instance, you know that negative numbers are needed to solve equations and to plot the graph of equations on a coordinate plane. So if a pupil cannot handle the concept that $-|x| \leq x \leq |x|$, you shall not worry too much about it.

Higher mathematics you learned here are certainly helpful. For instance, if you have a little knowledge on number theory, you know that the integer division $p \div s = q \ldots r$ with a quotient and remainder happens only for whole numbers, so you don't waste students' time and energy to consider the remainder of $7 \div (-2)$ and whether $-3$ is a factor of 21?

One more example. You know that a junior high student has to solve quadratic equations $ax^2 + bx + c = 0$, then you shall not worry too much if he cannot factorize $6x^2 + 5x - 6$. But you must insist that he know how to complete the square (平方完成). Because you know that polynomial factorization is not required frequently, and completing the square is going to be repeatedly required throughout the math curriculum. The purpose of polynomial factorization is never to find the roots, but it is required to do the antiderivatives (原始関数) of rational functions (有理関数) by the technique of partial fractions (部分分数). A junior high student is not in a hurry to master this technique.

Another type of context that will greatly enrich your professional career is the cultural context.

## Cultural Contexts 文化的脈絡

Although we have 九章算經 in China and you have Wazan (和算) in Japan, we have to admit that the modern mathematics is not in our culture. There is only one style of modern mathematics, namely, the western style, which is a Greek tradition. This is another reason that after some point, mathematics no longer looks like a native language but a foreign one. Given its Greek origin, it is a foreign language after all.

Just like learning a foreign language, it is always much helpful if you try to appreciate their culture at the same time.

The classics *Mathematics in Western Culture* by Morris Kline and *Gödel, Escher, Bach* by Douglas Hofstadter are great references for this topic. I believe you have them translated in Japanese. To study mathematics in the cultural contexts is fun by itself, and it also deepens your understanding of mathematics and provides plenty of stories and pictures to enrich your teaching content.

When you open up your eyes and be aware of, you will find mathematics really everywhere in western culture: in the literature, in the paintings and drawings, in music, in architecture of course, in graphical designs, in fashions, and in talk shows. Japan, after more than one hundred years of absorbing the western culture, has started to weave mathematics into artistic and humanity fabrics. A great example is 博士熱愛的算式 by 小川洋子, which I admire very much. And there are very interesting Math Rocks in Japan. I don't know for sure, but I think there are many Japanese resources of mathematical cultures for you to enjoy. So enjoy them, and have them come into line with your professions.

## Life-Long Learning 生涯学習

It is a challenge that comes with the merits and comforts of the new age, the 21st Century: Things you learned and skills you acquired in school are not expected to last a whole working life. People are expected to continue or to resume the status of a learner through a good part of their professional life.

In this sense, people are better advised to prepare themselves for a life-long learning when they are still in school (including colleges and universities). What kinds of knowledge and skills are about to facilitate the future learning, provided that we have no exact idea of what is going to happen and what will be there for us to learn?

A fundamental principle and a common advice are the following: Language, Language, and Language.

The first language is the natural languages, for instance, Japanese, English, or Chinese or anything else. My school has a policy to set a minimum requirement on English proficiency for every undergraduate student, the so-called graduation threshold (英文卒業標準). As the director of the Language Center, I held the responsibility to implement the policy and set the actual passing scores.

For this tough task, I tried to play fair. I put myself in the tests like TOEIC, IELTS, and TOEFL iBT. I did it as I was naturally—an educated person who uses English frequently but not on a daily basis—without any preparation. And I foretold the students that I am going to set 60 percent of my score to be the passing score for them. Do you think it is fair? I hope so.

It turned out that I got 960 in TOEIC and 116 in TOEFL iBT. So the 60-percent principle sets 570 in TOEIC and 68 in TOEFL iBT as the passing scores for students in the Math Department of NCU. And here I give you the reference.

The Language Centre at NCU has a branch of Teaching (Mandarin) Chinese for Speakers of Other Languages. It is a medium-sized program that is, from my point of view, good for a learning community: it's easy for all students and instructors to know each other, and staff members are able to give personal help to almost everyone. There are roughly 60 students for each term (four terms in a year), and roughly 5 of them are from Japan. When you have an opportunity to be an exchange student in the Math Department at NCU, I personally suggest you to take the chance to acquire some Chinese in our Chinese Language Program.

The second language is the formal language for relations and changes, forms and shapes, quantities and the measurement systems and their manipulations, and the description of data and uncertainties. It is the language that precisely describes and sometimes predicts all sorts of phenomena in Nature, Engineering, Human Society, Financing and Business Models.

You know what I am talking about. The second language that facilitates life-long learning is Mathematics. Congratulations, you are already good at it. Keep this in mind, learn mathematics to be a fluent and efficient reader and speaker, not necessarily as a writer (俳人), although when the situation comes you may become one.

The third language is the language of processing. It is always there throughout the human history. There are all kinds of procedures, or Algorithms, for all sorts of productions. However, the Tool of Production has been largely automated by machinery controlled by digital devices.

## Languages for Automation コンピューター言語

So we can say that the third language for life-long learning is a programming language in a general sense. I would like to call them Languages for Automation. What languages are there for us, the math majors, to acquire?

For mathematical text processing, I believe TeX or LaTeX or any variance in the family is the default. It is especially preferable for math teachers. TeX is a system on its own, and it is embedded in many word processors, notably Math Type with Microsoft Word and PowerPoint. Google document and a MathJax for web pages are catching up.

A side-benefit of writing in TeX/LaTeX is that it has set the standard of the so-called Mark-Up Language, the computer language that does the job of word processing in general. Most notable is the HyperText Markup Language that produces essentially the whole world of Web. I suggest math students learn TeX/LaTeX first, then continue to HTML, and that shall fulfill the purpose for any technical word processing and personal publishing for your career.

As the amount of data is growing up exponentially in this era, a database is inevitable to do the dirty job. It looks like an on-going project of handling big amount of data, but I think the classic Relational Database Management System is going to live with us, and the fundamental language for this system, SQL, will be the core of any language that handles database. The Relational Database is originally designed in terms of mathematical set theory. For instance, the JOIN operator, which is the soul of the Relational Database, is the Cartesian product of two sets (集合の直積). An efficient query that retrieve data from the Base is essentially a logical expression involving AND, OR, and NOT. We are supposed to be experts in these expressions.

And there are mathematical software. We shall be experts on them: who else can take on this job? All math software comes with an interface language for input commands interactively. But they can also run in the batch or script mode, so they can serve the purpose for mathematical programming. Matlab (and its clones) is the default for matrix computation. My colleagues teach Linear Algebra and Numerical Analysis along with Matlab. I also use Matlab to teach programming. Actually, Matlab is an excellent tool for statistical computations. However, statisticians prefer other software packages like SAS, S-plus, and R. I think you should be familiar with at least one of them.

The CAS, Computer Algebra System, may be the most fascinating math software. In return of the command 1/2 it gives $\frac{1}{2}$ instead of 0.5. And in case someone still thinks Calculus a human privilege, they show just the opposite. The big brand-names of CAS are Maple and Mathematica, and there is an open-source alternative, Maxima. I teach Calculus along with Maple or Maxima, and I think the courses of Algebra and Differential Equations can do the same.

For school teachers, software packages Geogebra and GSP (Geometers' Sketch Pad) are handy. Geogebra handles most topics in highschool algebra, analytical geometry, and basic analysis well, and GSP is great for classic plane geometry.

As for the programming language in general use, I believe C and FORTRAN will not die easily. However, if you are not going to be a serious scientific computing expert, you can skip FORTRAN. Anyone who is serious about computer science must read and write C. Like it or not, C is the common core language for all programmers and it is the prototype for all programming languages.

What language shall you bet beyond C? I really don't know. I did not like Java from the very beginning, and I am glad to see it perish nowadays. At this moment, I think Python is promising. But I don't know for sure, I am going to teach Python for the first time next semester, and I will make a decision if it is the last language I will learn before I retire.

To conclude this section, I would like to say that, according to my personal experience over the past 30 years, most computer software become trash very quickly. Most of the time, pursuing the newest and hottest computer trend is a waste of your life. We mathematicians have a trained talent to distinguish the hype and the core, and I suggest you to use this special talent carefully to retrieve true information from messages with lots, lots of background noise.

## ALL are Numbers　すべてはナンバー

Mathematics, Computer Science, and Informatics are different professions. Math majors are not supposed to be their substitutes. We are not trained to become computer scientists. However, math majors are on a cutting edge of the world of digital informatics. The fundamental reason of my arguments is that, in a computer, all are numbers: everything is presented in numbers and there is nothing but numbers.

To prepare yourself for entering the world of informatics, I think you can start with the fundamental concepts of computers. Among those fundamentals, what matters most for us are the data representations in numbers.

Characters are coded in numbers, for instance Chinese, Japanese, and Korean characters were coded in the CJK code-set which now becomes a part of Unicode (utf-8).

The glyphs, the actual symbols you see on the screen that you recognize as characters and words, are a special kind of digital graphs which are basically either binary matrices or coefficients of splines (third degree polynomial functions).

Colors you see on the output device are four dimensional vectors: red, green, blue, and the so-called alpha channel. A picture is a matrix of colors. A moving picture (video) is a time series of pictures. There are two basic techniques to compress the big amount of data in videos: One based on the graph theory and the other on real analysis (calculus).

Audio signals, including voices and music and unheard radio waves, are simply sequences of numbers. In the core sits the Shannon sampling theorem , and the compression is now done by Fourier or wavelet analysis.

I think you get my point, and I shall refrain from going into endless details. The bottom line is that in a computer, all are numbers. And we are good at numbers.

## Math Curriculum 数学カリキュラム

The curriculum in the Mathematics Department will generally be enough to prepare a student for an expertise in Computer Science or Informatics. And I would like to make some comments on the curriculum. First of all, I think a 3- or 4-credit course of Fundamental Concepts of Computers will help much to put all essential things together in one place. My department has been doing this for more than 10 years.

The Fundamental Concepts of Computers, together with Calculus (微積分), Statistics (統計), and Linear Algebra (線形代数), shall become the four corners of the solid foundation of a math curriculum. We talked very little about statistics simply because of time constraints. There are voices in the West that argue the statistics shall be taught before calculus. But we will put this topic aside for the present.

We all know calculus. Allow me to add just one comment on calculus. It is the mathematics for the continuum as is known in the sense that time and space are continuum (連続体). Many realistic objects are not continuum, for instance money and population are discrete, and mechanical systems are usually discrete. In a certain sense fluids like air and water consist of discrete particles. Even light has a particle model. However, when we do not have a tool to deal with such discrete objects with a huge amount of quantity, we can pretend they are continuous and design smooth mathematical models for them, and manipulate these models by calculus. Calculus has served successfully as a super calculator for the past 300 years. And calculus helped the invention of computers.

So here come the computers. Objects of discrete nature have a chance to be handled directly in a discrete mathematical model. New mathematical tools for discrete model are developing fast now, but the ultimate theory is yet to come. Students can acquire a good amount of recently developed knowledge in the courses of Discrete Mathematics (離散数学), Finite Mathematics (有限数学), and Algorithms (アルゴリズム). And people usually think Linear Algebra is the common foundation of these mathematics in a discrete style.

But Linear Algebra comes in two flavors: pure and applied. The common core of both is the fact that some matrices are essentially the same: They are the same linear transformations represented in different bases. Let's call them similar matrices. The theme of a Linear Algebra course in pure flavor is to identify the invariants of similar matrices, and to determine a unique canonical representative for any class of similar matrices. The theme in applied flavor is to convert similar matrices from one form to another as efficient and stable as possible. Students who are preparing for the computational or informatics professions are advised to have a quick understanding of the pure side, and devote much of their time to the applied side.

## Epilogue 結語

To wrap things up, I suggest an approach of thinking about mathematics education: to think by making an analogy with the learning and teaching and purposes of languages. This approach can be applied to preparing your own education, and it is also helpful when you become a math teacher and take the responsibility for the math education of other people. Mathematics, just like languages, sit in the core of our civilization. That is why mathematics is found in almost everything around us. Although for the time being, we talked mainly on the relations with informatics, math majors must have confidence that you are versatile by default. You just have to seize the chance to educate yourself right here at the Math Department in Ryukoku University.

I hope there are useful hints for you, and finally I have best wishes for you and thanks your attention.