Quadrature Rules Needed in Galerkin-Wavelets Methods*

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Abstract

In this article we will review several numerical quadrature rules, their theoretical background and implementations, that are needed in a discretization of Galerkin type for differential equations, using wavelets as basis functions. Material given here will be a summary of results of the recent three years by several groups of researchers in this field, including the author himself.

1 Introduction

There are several versions of definition for *wavelet.* People seem to agree that it is a function (sometimes called a *mother wavelet*) whose scaled dilations and translations form an orthonormal basis for $L^2(\mathbb{R})$. A popular form looks like $\sqrt{2^j}\psi(2^jx-k)$, where $(j,k) \in {}^2$. Yet the dilation factor 2 and the integer translation are not necessary. When people construct wavelets, they also keep in mind to have the energy localized in both time and frequency domains (although it cannot be exponentially decay simultaneously). In many applications, including ours, the degree of approximation abilities is also crucial. Some think the orthogonality on the same level j is not essential and drop it to study *pre-wavelets.* And some construct a basis from several families of wavelets, or *wavelet packets.*

The idea of wavelets may be traced back to Calderon [7] and Coifman and Weiss [14]. Modern ideas grow out of applications in signal analysis, cf. Liénard [26] and Goupillaud, Grossmann and Morlet [21]. Earlier mathematical analysis of wavelets can be found in Stromberg [35] (unnoticed until 1988), Grossmann and Morlet [22] (in which the name of *wavelets* first appeared) and Meyer [30]. In recent years, the multiresolution approximations of $L^2(\mathbb{R})$ has been developed by Mallat [28] and Meyer [32] to establish a framework for the construction of wavelets. The multiresolution approach has lead Daubechies [16] to the discovery of a family of compactly supported wavelets. This family includes the Haar function [23] as a special case, which was not considered as a wavelet by earlier definitions because the lack of Utilizing the same approach, Chui and Wang [9, 10] found another framework of constructing wavelets and pre-wavelets from B-splines. Battle [4] proposed yet another approach based on renormalization groups in the study of physical systems with infinitely many degrees of freedom. regularities.

Applications of wavelets are better established in the fields of signal processing, image processing and data compression. Basic references should point to the works of Mallat [29], Daubechies [17], and Coifman, Meyer, Quake, and Wickerhauser [12]. The group of Coifman has a package of computer software, with graphical user interface, which implements a few algorithms [13]. The executable codes for several systems are currently released in public domain. See also the last

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section for public resources on computer networks. Besides, there are applications to statistics [19], quantum mechanics and the solution of integral equations [1].

Because of their similarities with Fourier transforms and finite elements, wavelets are studied for the applicability of being a tool in scientific computation and numerical analysis, such as the numerical solution of differential equations. Among the recent articles on the later subject, we refer to the works by Bacry, Mallat and Papanicolaou [3], Beylkin, Coifman and Rokhlin [6], Glowinski, Lawton, Ravachol and Tenenbaum [20], Jaffard [24], Maday, Perrier and Ravel [27], and Xu and Shann [37].

In the application of wavelets to the numerical solution of differential equations, it is usually easier with those discovered by Daubechies [16], which have compact supports and form an orthonormal basis of $L^2(\mathbb{R})$. In this case, the *wavelet* $\psi(x)$ is derived from the *scaling function* $\phi(x)$ which is normalized to have $\int \phi = 1$ and satisfies the *scaling equation*

$$\phi(x) = \sum_k c_k \phi(2x-k). \tag{1.1}$$

The coefficients c_k were computed by constructing a certain trigonometric polynomial $L(\xi) = \frac{1}{2}\sum c_k e^{-ik\xi}$, see Daubechies' "Ten Lectures" [18].

The construction of these scaling and wavelet functions is not the topic of this article. Interested readers will find accessible information from Strang's review article [34] or the NSC Project Report by Chern, Chou, and Shann [8]. The scaling coefficients of Daubechies' wavelets of orders $1 \le p \le 14$ were computed by the author's group in quartic precision on a VAX 6510.

Once $\phi(x)$ is determined, the multiresolution approach asserts that

$$\psi(x) = \sum_k (-1)^k c_{1-k} \phi(2x-k) \tag{1.2}$$

will be a wavelet. One of the properties of ψ is that $\int x^m \psi \, dx = 0$ for $0 \le m \le p-1$.

Define

$$\phi_{jk}(x) := \sqrt{2^j}\phi(2^j x - k) \quad \text{and} \quad \psi_{jk}(x) = \sqrt{2^j}\psi(2^j x - k). \tag{1.3}$$

Let $W_j = \operatorname{span}\{\psi_{jk}(x) \mid k \in \}$. Then $\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R})$. Let $V_j = \bigoplus_{\ell=-\infty}^{j} W_\ell$. Since we do not want to deal with both infinities, we usually let

$$V_J = V_{-1} \oplus W_0 \oplus \cdots \oplus W_J, \quad \text{for } J \ge 0.$$

Numerical computations are carried out for a sufficient large J. If $\psi(x)$ is related to a multiresolution approximation of L^2 (all wavelets in use are indeed related to one), then there is a *scaling function* $\phi(x)$ such that $\{\phi_{jk}(x) \mid k \in \}$ forms an orthonormal basis for V_j. Thus there are two kinds of basis for V_J. The first involves both ϕ and ψ and we add higher levels of ψ for higher accuracy; the second involves ϕ only and we construct new set of basis for higher accuracy. Both are usable in different situations. For $p = 1$, $\phi(x) = \chi_{[0,1]}$; for $p = 2$, it is sometimes called $D_4(x)$ to name after Daubechies. There is no elementary function expression for this new function.

Wavelets that "live" in intervals can be derived from those "live" on real lines by, basically, Gram-Schmidt process for those crossing the boundary. They are first constructed by Meyer [31] then improved by Cohen, Daubechies, Jawerth and Vial [11] and Andersson, Hall, Jawerth and Peters [2]. Yet no experiments have been seen for these wavelets.

Approximation properties of the finite dimensional subspaces from which we choose the trial functions is a crucial part of the error analysis in Galerkin methods. For the family of Daubechies' wavelets, Xu and Shann [37] proved that, for $u \in H^s$, $0 \leq s \leq p$, and $P_J u$ being the L^2 projection of u into V_J, we have

$$\|u - P_J u\| \leq C 2^{Js} |u|_s, \tag{1.4}$$

for some positive constant C independent of J and u. Here p is the order of wavelets being used. The technique is the Bramble-Hilbert lemma commonly applied in finite elements and the key point is the decay of wavelet coefficients, $|(u, \psi_{jk})| = O(2^{-js})$.

2 State of the problem

When wavelets are applied to solving differential equations in the framework of the Galerkin methods (or the more general methods of minimal residual), numerical quadrature rules are the fundamental construction blocks. Following the work by Beylkin, Coifman and Rokhlin [5], there are one-point and multi-point quadrature rules. The *one-point rules* are based on the Taylor's expansion and the *multi-point rules* are based on the polynomial interpolations. The former has a fixed, but very attractive, order; the latter may have higher order if more points are used.

For the multi-point rules, we need the values of quadratures involving polynomials. We will use scaling equations to derive formulae of quadratures involving polynomials and scaling/wavelet functions with compact supports. It turns out that with a few parameters, which are theoretically exact, these quadratures can be evaluated with algebraic formulae instead of numerical approximations. Those parameters can be obtained with high precision by solving well-conditioned linear systems of equations which involve matrices already seen in the literature of wavelets for other purposes.

In the discretization of a differential equation by Galerkin methods, the fundamental building blocks of the stiffness or mass matrices are the L^2-inner products of the coefficients and the basis functions, in our case scaling and wavelet functions.

For a linear problem, we will be involved with the integrals of the following forms:

$$\int f(x)\, \phi_{jk}(x)\, dx \tag{2.1}$$

$$\int f(x)\, \phi_{jk}(x)\, \phi_{\ell m}(x)\, dx \tag{2.2}$$

$$\int f(x)\, \phi'_{jk}(x)\, \phi_{\ell m}(x)\, dx \tag{2.3}$$

and

$$\int f(x)\, \phi'_{jk}(x)\, \phi'_{\ell m}(x)\, dx \tag{2.4}$$

In each instance of ϕ, one may replace it with ψ. Recall that (2.1) is needed for the right-hand-side vector; and are needed for the construction of stiffness and mass matrices respectively; and will appear if the problem is nonsymmetric.

If the problem is nonlinear, we will have even longer integrands like

$$\int f(x)\,\phi_{jk}\,\phi_{lm}\,\phi_{st}\,dx \tag{2.5}$$

and

$$\int f(x)\,\phi_{jk}\,\phi_{lm}\,\phi_{st}\,\phi_{uv}\,dx. \tag{2.6}$$

Similarly, each instance of ϕ may be replaced by ψ or their derivatives.

3 Reduction of computations

Reducing the computations involving wavelets to the manipulation of c_k is a general strategy. It is sought for every kind of application for wavelets. Examples have been seen from the plotting of $\phi(x)$ and $\psi(x)$ and their derivatives (in the classical sense) to numerical algorithms [5] and signal decomposition [28], among others.

There seems to have a lot of cases in –(2.6). We will reduce these forms to some simple ones. They are not only theoretically feasible, but also computationally implementable.

The first reduction is to assert that only rules for ϕ's are necessary. It is because of the construction of ψ from ϕ in (1.2). Clearly all instances of ψ_{jk} can be replaced with $\phi_{\ell m}$.

Then we observe that, if $\phi \in C^s$, $\phi^{(s)}(x)$ also satisfies a scaling equation similar to (1.1). That is,

$$\phi^{(s)}(x) = \sum_k 2^s\, c_k \phi(2x-k). \tag{3.1}$$

Thus the techniques involving (1.1) that are applied to the quadratures without the derivatives can be applied to those with derivatives in general. For instance, we will see that they correspond to eigenvectors associated with different eigenvalues. Therefore, at this moment, we shall omit the cases with derivatives; and come back to them after the opposite cases are resolved.

For (2.1), after a proper change of variable we have the integral

$$\int f(x)\phi(x)\,dx. \tag{3.2}$$

For (2.2), we should first check if ϕ_{jk} and $\phi_{\ell m}$ have nonzero common supports. Note that the support of $\phi_{jk}(x)$ (and its derivatives) is

$$[\frac{k}{2^j}, \frac{\ell+k}{2^j}].$$

It is easy to implement a logical function to check for this condition.

If there is a nonempty intersection, we have some real work to do. We will first bring the level of $\phi(x)$ up to j (where $j=0$ is a special case). This can be done by recursively apply the scaling equation (1.1). In general, if we already have

$$\phi(x) = \sum_{n=0}^{N_{j-1}} \alpha_{j-1,n}\phi(2^{j-1}x-n) \in V_{j-1},$$

we can represent the same function in V_j by applying (1.1) to each $\phi(2^{j-1}x-n)$. Formally we have

$$\phi(x)=\sum_m\left(\sum_n \alpha_n c_{m-2n}\right)\phi(2^jx-m)\in V_j.$$

It can be written in a vector-matrix language:

$$\begin{pmatrix}\alpha_{j-1,0} & \alpha_{j-1,1} & \cdots & \alpha_{j-1,N}\end{pmatrix}\begin{pmatrix} c_0 & c_1 & \cdots & c_\ell & & & & & & \\ & & c_0 & c_1 & \cdots & c_\ell & & & & \\ & & & & c_0 & c_1 & \cdots & c_\ell & & \\ & & & & & \ddots & & & & \\ & & & & & & c_0 & c_1 & \cdots & c_\ell \end{pmatrix}_{N+1\times(2N+\ell+1)}$$

The above multiplication can be implemented in an $O(N)$ algorithm. In fact, since the representation of $\phi(x)$ in V_j will be used frequently in a project, it makes more sense to compute and store all of these values for a table-lookup.

With this change-of-level, we can move all the levels up to the highest one. In particular, with a proper change of variable, (2.2) can be reduced to

$$\int f(x)\phi(x)\phi(x-k)\,dx. \tag{3.3}$$

Similarly, (2.5) and (2.6) are reduced to

$$\int f(x)\,\phi(x)\,\phi(x-k)\,\phi(x-m)\,dx \tag{3.4}$$

and

$$\int f(x)\,\phi(x)\,\phi(x-k)\,\phi(x-m)\,\phi(x-s)\,dx. \tag{3.5}$$

Note that this technique is also applicable, with little modification, to $\phi^{(s)}$ because of (3.1).

A general strategy is recommanded as follows. If $f(x)$ is a polynomial, raise ϕ's at different levels to the highest one, change variable and find the values by methods in Sections 5–7. In fact for the common used wavelets, those values should be stored. If $f(x)$ is a general smooth function, apply the one or multi-point rules in Sections 4 and 8. Since the accuracy depend on the length of common supports of the involved ϕ's (that is, the width of the integration limits), we may need to raise the level high enough before the rules are applied. Then the original integral at a lower level can be reconstructed by a fast pyramid algorithm based on the scaling equation (1.1).

4 The one-point rule

The one-point rule in principle can be applied to (2.1) and (2.2). However, there is a special advantage on its accuracy for the integral (2.1).

We do not reduce (2.1) to (3.2) when the one-point rule is applied. Conversely, we try to raise the level j to high enough; that is, the suppϕ_{jk} is small (like $O(2^{-j})$). By the scaling equation (1.1) we can reconstruct an integral with low level ϕ by a fast transform from the integrals with high levels. The computational complexity is as usual depends on the "mesh size" (now $O(2^{-j})$) and number of function evaluations (now $O(2^j)$).

The family of *one-point* rules are derived from the Taylor expansion of $f(x)$, if exist. Since

$$\int f(x)\phi_{jk}(x)\,dx = \frac{1}{\sqrt{2^j}}\int f(\frac{x+k}{2^j})\phi(x)\,dx.$$

We expand f centered at the point $\frac{k+\tau}{2^j}$:

$$f(\frac{x+k}{2^j}) = f(\frac{k+\tau}{2^j}) + \sum_{n=1}^{N-1}\frac{1}{n!}f^{(n)}(\frac{k+\tau}{2^j})(\frac{x-\tau}{2^j})^n + \frac{1}{N!}f^{(N)}(\xi_x)(\frac{x-\tau}{2^j})^N$$

If

$$\int (x-\tau)^n\phi(x)\,dx = 0, \quad \text{for } 1 \le n \le N-1,$$

then we have the one-point quadrature rule

$$\int f(x)\phi_{jk}(x)\,dx \approx \frac{1}{\sqrt{2^j}}f(\frac{k+\tau}{2^j}); \tag{4.1}$$

with the approximation error being

$$\frac{1}{N!}2^{-j(N+1/2)}\int f^{(N)}(\xi_x)(x-\tau)^N\phi(x)\,dx. \tag{4.2}$$

For the original scaling function of order p by Daubechies, the best choice is to take $\tau = M_1^p := \int x\phi\,dx$. (See Section 5.) For each p there can be only one vanishing "shifted" moment. Thus we have $N = 2$ in (4.2) and (4.1) is a second order quadrature rule. However, Sweldens and Piessens [36] found that, for $p \ge 2$, actually

$$M_2^p = (M_1^p)^2 .$$

Therefore

$$\int (x - M_1^p)^2\phi(x)\,dx = M_2^p - 2(M_1^p)^2 + (M_1^p)^2 = 0.$$

Hence, with the choice $\tau = M_1^p$, the one-point quadrature rule (4.1) is actually a third order scheme.

However, computations show that $M_1^p \approx 0.2(p+1)$, which is roughly a fixed number around 1. Then (4.2) indicates that, when j is not large enough but $|k|$ is large ($\phi_{jk}(x)$ is far away from the "center"), the constant term in the error may be severe. A mathematical translation may be necessary before performing the numerical quadrature rules.

Answering the request of Coifman, Daubechies designed another set of compactly supported wavelets and scaling functions, now called *coiflets* by some colleagues. For each $p \ge 2$, there are $3p$ nonzero scaling coefficients for the coiflet. In addition to the properties from the original set of wavelets, the coiflets have

$$\int (x-\tau_p)^n\phi(x)\,dx = 0, \quad \text{for } 1 \le n \le p-1.$$

Coefficients $c_0, \ldots, c_{3p-1}$ can be solved by the original set of equations plus the following set:

$$\sum_k c_k(k-\tau_p)^m = 0, \quad \text{for } 1 \le m \le p-1.$$

Numerical values of c_k together with τ_p, for $p = 2$, 4 and 6 are listed in [5].

If coiflets are chosen for the numerical computations, the one-point quadrature rules (4.1) are of order p. For $p \ge 4$, they have an advantage to the original Daubechies' wavelets. However, the cost is at least a 50% increase of operation counts.

5 Integrals with polynomials

We now consider the case of (3.2) when $f(x)$ is a polynomial. In particular, we need the following parameters:

$$M_m^p = \int x^m \phi(x)\, dx$$

A number of colleagues [15, 36, 33] have found the following identity independently.

$$M_m^p = \frac{1}{2(2^m-1)} \sum_k c_k \sum_{\ell=1}^{m} \binom{m}{\ell} M_{m-\ell}^p\, k^\ell \tag{5.1}$$

This recursion relation starts from $M_0^p = \int \phi\, dx = 1$, for any $p \geq 1$. The recursion (5.1) should not be confused with that by Beylkin *et al.* [5, p. 179]; the former is a "closed form" formula and the later defines a convergent sequence.

As an immediate application, we can construct the polynomials determined by

$$P_m(x) = \sum_k k^m \phi(x-k).$$

For $0 \leq m \leq p-1$, we have

$$\begin{aligned} x^m &= \sum_k M_{m,k}^p \phi(x-k) \\ &= \sum_k \Big(\sum_{\ell=0}^{m} \binom{m}{\ell} k^\ell M_{m-\ell}^p \Big) \phi(x-k) \\ &= \sum_{\ell=0}^{m-1} \binom{m}{\ell} M_{m-\ell}^p \sum_k k^\ell \phi(x-k) + \sum_k k^m \phi(x-k). \end{aligned}$$

Thus

$$P_m(x) = x^m - \sum_{\ell=0}^{m-1} \binom{m}{\ell} M_{m-\ell}^p P_\ell(x).$$

Since $P_0(x) = 1$, we see that $P_m(x)$ is a polynomial of degree m. Conversely, given any polynomial $q(x)$ of degree $m \leq p-1$, there is a polynomial $r(x)$ of the same degree that $q(x) = \sum r(k)\phi(x-k)$.

We now proceed to a digression on the integrals with fixed bounded limits. In this case the quadrature values are reduced to linear algebraic problems. We illustrate the situation by the integrals

$$N_{m,k}^p = \int_0^\infty x^m \phi(x-k)\, dx.$$

For the translation parameters $k \leq 1-2p$, the integrals are zero. For $k \geq 0$, integration over $[0,\infty)$ is the same as over $\mathbb{R}$. Thus $N_{m,k}^p = M_{m,k}^p$. We now concentrate on the cases that $2-2p \leq k \leq -1$.

Again, we start from (1.1) with a changing of variable:

$$\int_0^\infty x^m \phi(x-k)\, dx = \frac{1}{2^{m+1}} \sum_\ell c_\ell \int_0^\infty x^m \phi(x-(2k+\ell))\, dx. \tag{5.2}$$

If $1-p \le k \le -1$; that is, over half of the support is still inside of $[0,\infty)$. Then the supports of $\phi(x-(2k+\ell))$, for $-2k \le \ell \le 2p-1$, actually lies entirely in $[0,\infty)$. That means, for these ℓ, the integrals on the right hand side of (5.2) can be evaluated as in Section 3. Over all, we have

$$N^p_{m,k} = \frac{1}{2^{m+1}}\Big[\sum_{\ell<-2k} c_\ell N^p_{m,2k+\ell} + \sum_{\ell\ge -2k} c_\ell M^p_{m,2k+\ell}\Big].$$

Suppose the part of $\ell \ge -2k$ is known, we have a linear system in $N^p_{m,k}$.

Given a $p \ge 2$ (there is nothing to do with $p=1$) and fix an $m \ge 0$, we have unknowns $N^p_{m,k}$ for $2-2p \le k \le -1$. Now let

$$\begin{cases} x_k = N^p_{m,-k}, \\ b_k = \dfrac{1}{2^{m+1}} \displaystyle\sum_{\ell=2k}^{2p-1} c_\ell M^p_{m,\ell-2k}, \end{cases} \qquad k = 1,2,\ldots,2p-2.$$

Let $\boldsymbol{b} = (b_k)^T$ and $A = (a_{k\ell})$ be the matrix of order $2p-2$ such that $a_{k\ell} = c_{2k-\ell}$. Then $\boldsymbol{x} = (x_k)^T$ is the solution of the system

$$(I - \frac{1}{2^{m+1}}A)\,\boldsymbol{x} = \boldsymbol{b}.$$

Note that A is independent of m and it is a finite submatrix of the low pass filter matrix L defined in [34], in which it is also shown that A has an eigenvalue $\lambda = 1$ with an associated eigenvector $(\phi(1),\phi(2),\ldots,\phi(2p-2))^T$. Here we claim another property for A.

Proposition 5.1. *Let A be the matrix defined above. $\|\cdot\|_2$ be the matrix norm induced by the Euclidean norm. We have*

$$\|A\|_2 \le \sqrt{2}.$$

Thus, for any $m \ge 0$ and $p \ge 2$, the *Neumann series*

$$\sum_{n=0}^{\infty}(\frac{1}{2^{m+1}}A)^n \tag{5.3}$$

is an efficient method for computing the inverse of $I - \frac{1}{2^{m+1}}A$. We have prepared values for $2 \le p \le 14$ and $0 \le m \le 14$. The iterations are computed in quartic precision with the stop criterion being that the ℓ^2-norm of error $< 10^{-24}$.

6 Linear systems of equations

In the last section we have seen a conversion of integration problem into a linear system of equations which involve only with the scaling coefficients c_k of the wavelets. In fact this is a common situation for other similar kinds of integrals. Let us start with (3.3) with polynomials:

$$L^p_{m,k} = \int x^m \phi(x)\phi(x-k)\,dx$$

Since $L^p_{0,k} = \delta_{0k}$, we start from $m=1$. The support of $\phi(x)$ of order p is $[0,2p-1]$, hence for each $p \ge 1$ and $m \ge 1$, we have unknowns $L^p_{m,k}$ for $2-2p \le k \le 2p-2$. Then

$$\begin{aligned} L^p_{m,k} &= \int x^m \sum_n c_n\phi(2x-n)\sum_\ell c_\ell\phi(2x-2k-\ell)\,dx \\ &= \tfrac{1}{2}\textstyle\sum_{n,\ell} c_n c_\ell \int (\frac{x+n}{2})^m \phi(x)\phi(x+n-2k-\ell)\,dx \\ &= \frac{1}{2^{m+1}}\Big[\sum_{n,\ell} c_n c_\ell L^p_{m,2k+\ell-n} + \sum_{n,\ell} c_n c_\ell \sum_{r=1}^{m}\binom{m}{r} n^r L_{m-r,2k+\ell-n}\Big]. \end{aligned} \tag{6.1}$$

The problem is recursively equivalent to a system of linear equations

$$(I - \frac{1}{2^{m+1}}A)\boldsymbol{x} = \boldsymbol{b}. \tag{6.2}$$

Let $N = 2p - 2$, then $\boldsymbol{x} = (L^p_{m,-N}, \ldots, L^p_{m,N})^T$, $\boldsymbol{b} = (b_k)$ and $A = (a_{kj})$ with

$$b_k = \frac{1}{2^{m+1}} \sum_{n=1}^{2p-1} c_n \sum_{\ell=\ell_1}^{\ell_2} c_\ell \sum_{r=1}^{m} \binom{m}{r} n^r L^p_{m-r,2k+\ell-n}, \quad \text{where } \begin{cases} \ell_1 = \max\{n-2k-N, 0\}, \\ \ell_2 = \min\{n-2k+N, 2p-1\} \end{cases}$$

and

$$a_{kj} = \sum_n c_n c_{n+j-2k}, \qquad -N \le j, k \le N.$$

This matrix A was considered by Lawton [25] for a sufficient and necessary condition on c_k (and hence $m_0(\xi)$) for constructing orthonormal wavelets basis. It is noted that A has an eigenvalue 2 with an associated eigenvector $\boldsymbol{v} = (\delta_{0k})$. The condition is that 2 is a nondefective eigenvalue of A. Daubechies [18] summerized it as one of the three equivalent conditions.

Since $\phi(x)$ is infinitely many times differentiable as $p \to \infty$, differentiate (1.1) to have

$$\phi^{(s)}(x) = 2^s \sum_k c_k \phi^{(s)}(2x - k).$$

Apply this equation into $\int \phi(x)\phi^{(s)}(x-k)\,dx$ and repeat the steps in (6.1), we find that the matrix $A = A(p)$ will asymptotically have eigenvalues 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, ..., as $p \to \infty$.

Actually A has a simple pattern. On the even-numbered columns, by , all entries are 0 except $a_{k,2k} = 2$. On the odd-numbered columns all entries are 0 except for $\lceil \frac{j-2p}{2} \rceil \le k \le \lfloor \frac{j+2p}{2} \rfloor$, which is the $2p$-element vector

$$\boldsymbol{h} = \begin{pmatrix} \sum_k c_k c_{k+(2p-1)} \\ \vdots \\ \sum_k c_k c_{k+3} \\ \sum_k c_k c_{k+1} \\ \sum_k c_k c_{k+1} \\ \sum_k c_k c_{k+3} \\ \vdots \\ \sum_k c_k c_{k+(2p-1)} \end{pmatrix} = \begin{pmatrix} \int \phi(x)\phi(x - \frac{2p-1}{2})\,dx \\ \vdots \\ \int \phi(x)\phi(x-\frac{3}{2})\,dx \\ \int \phi(x)\phi(x-\frac{1}{2})\,dx \\ \int \phi(x)\phi(x-\frac{1}{2})\,dx \\ \int \phi(x)\phi(x-\frac{3}{2})\,dx \\ \vdots \\ \int \phi(x)\phi(x-\frac{2p-1}{2})\,dx \end{pmatrix}.$$

Proposition 6.1. *Let A be the matrix defined in this section. $\|\cdot\|_2$ be the matrix norm induced by the Euclidean norm. We have*

$$\|A\|_2 \le 2\sqrt{2}.$$

Fortunately the linear system (6.2) is not needed for $m = 0$ (actually $\boldsymbol{b} = \boldsymbol{0}$ and $I - \frac{1}{2}A$ is singular). Hence for all cases with $p \ge 2$ ($p = 1$ is an empty case) and $m \le 1$, equation (6.2) can be solved by the Neumann series (5.3).

7 Eigenproblems

By (3.1), the computations of

$$\int x^m \phi'(x)\phi(x-k)\,dx \quad \text{and} \quad \int x^m \phi'(x)\phi'(x-k)\,dx$$

can be done essentially the same way as that for $L^p_{m,k}$. The exception is in the case of $m = 0$. Here we do not have the orthogonality of the translates of ϕ. For simplicity of presentation, let us abuse the notations and define

$$M_k := \int \phi'(x)\phi(x-k)\,dx.$$

First of all,

$$\int \phi'(x)\phi(x)\,dx = \int \phi(x)\,d\phi(x) = -\int \phi'(x)\phi(x)\,dx.$$

Thus $M_0 = 0$. Similarly we have $M_k = -M_{-k}$. And by the compact supportness, we only have to compute M_k for $1 \le k \le \ell - 1$.

It is easy to check that M_k form an eigenvector of A associated with eigenvalue 1, where A is the matrix defined in (6.2).

It is interesting to see how the eigenvectors are normalized? We do not know the norm of the wanted eigenvectors. The technique used by Strang [34] to calculate the dyadic values of scaling functions do not work in this case. Because there the entries in the eigenvector sum up to 1 ($\sum_k \phi(x-k) = 1$) but here of entries in the eigenvectors sum up to 0 ($\sum_k M_k = 0$).

By the remark after (5.1), we have

$$\sum_k k^m \phi(x-k) = \text{a monic polynomial of degree } m,$$

for $0 \le m \le p-1$. Thus, for $p \ge 2$ (this is not a restriction in our case, since the C^1 wavelets are having orders $p \ge 3$), we have $\sum k\phi(x-k) = x + b$. Multiply $\phi'(x)$ on both sides and perform an integration by parts on the right hand side, we have

$$\sum_k kM_k = -1.$$

For the integral with two derivatives,

$$N_k := \int \phi'(x)\phi'(x-k)\,dx,$$

we apply the same trick to normalize the eigenvector. Now it is associated with eigenvalue 1/2. That is, for $p \ge 3$ (this is not a restriction either), we have $\sum_k k^2\phi(x-k) = x^2 + ax + b$. Differentiate the equation once and multiply $\phi'(x)$ on both sides, perform the integration by parts will give us the desired formula

$$\sum_k k^2 N_k = -2!.$$

When the values for $m = 0$ are determined, the integrals with $m > 0$ can be solved by linear systems similar to (6.1). But, for smaller m, the Neumann series methods may not be applied.

However, there is a problem with existence and uniqueness of the required eigenvalues of the matrix A in (6.2). This problem is asserted in general (for wavelets of compact supports) by

Dahmen and Micchelli [15], with a tool from the field of approximation theory (in fact, developed before the wavelets appeared). Essentially they say that if the scaling function is differentiable then the eigenvalues in question exist and are unique. In their theory the scaling coefficients are the elements of a *mask* and the scaling equation is implemented by a *subdivision operator*. Daubechies' cascade algorithm is corresponding to their convergent *stationary subdivision scheme*.

An extension of this eigenproblem to the cases like (3.4) and (3.5) is also derived in [15], we will see in the last section.

8 The multi-point rules

The preparation of the integrals with $f(x)$ being polynomials makes more sense when they are applied to determine the parameters for multi-point quadrature rules by Beylkin, Coifman and Rokhlin [5]. This scheme is derived from the idea of polynomial interpolations.

Let us compute, for each p and M, the quadrature coefficients α_m^p such that

$$M_\ell^p = \sum_{m=0}^{M-1} \alpha_m^p m^\ell, \quad \text{for } 0 \le \ell \le M-1 \tag{8.1}$$

(define $0^0 = 1$). For these coefficients,

$$\int x^\ell \phi_{jk}(x)\, dx = \frac{1}{\sqrt{2^j}} \sum_{m=0}^{M-1} \alpha_m^p \left(\frac{k+m}{2^j}\right)^\ell, \tag{8.2}$$

for any $0 \le \ell \le M-1$. That is, the approximation

$$\int f(x)\phi_{jk}(x)\, dx \approx \frac{1}{\sqrt{2^j}} \sum_{m=0}^{M-1} \alpha_m^p f\left(\frac{k+m}{2^j}\right) \tag{8.3}$$

is *exact* for any polynomial of degree $\le M-1$. (There is a mistake in the original paper.)

Actually the formula (8.3) chooses nodal points $x_m = 2^{-j}(k+m)$, for $0 \le m \le M-1$. Let $p_M(x)$ be the interpolation polynomial of degree $\le M-1$ for $f(x)$ at these nodal points, then

$$f(x) = p_M(x) + \frac{1}{M!} f^{(M)}(\xi_x) \prod_{m=0}^{M-1} (x - x_m),$$

if $f \in C^M(\text{supp}\phi_{jk})$. Therefore the error of the quadrature rule (8.3) is

$$2^{-j(M+1/2)} \frac{1}{M!} \int f^{(M)}(\xi_x) \prod_{m=0}^{M-1} (x - x_m) \phi(x)\, dx. \tag{8.4}$$

Here ξ_x is assumed to have properly modified with the change of variables. Hence (8.3) is an order M method.

Note that the multi-point rules (8.3) do not reduce to the one-point rule (4.1) when $M = 1$, which is

$$\int f(x)\phi_{jk}(x)\, dx \approx \frac{1}{\sqrt{2^j}} f\left(\frac{k}{2^j}\right)$$

(note that $c_0 = \int \phi(x)\, dx = 1$). This is a first order method instead of a third-order method for the one-point rule. Theoretically, multi-point methods of this kind must have $M \ge 4$ to perform better than the one-point rule.

Recall that

$$\text{supp}\phi_{jk} = [2^{-j}k, 2^{-j}(k + 2p - 1)].$$

There are two problems with the nodal points for (8.3). First, if $M > 2p$, the points x_m might fall out of the support of ϕ_{jk}. For some problems $f(x)$ may not be defined out there. Choices are either an extrapolation or a lower order method. Secondly we note that, for smaller M, the nodal points x_m are gathered on the left hand side of the support. This need not be a severe problem since the interpolation error is larger near the right hand side of $\text{supp}\phi_{jk}$ where $\phi_{jk}(x)$ itself is usually very small.

However, Sweldens and Piessens [36] have addressed both of the problems. On the one hand, they shorten the distance between nodal points to raise the order M for a fixed p, while keeping all points within the support of ϕ_{jk}. On the other hand they introduce one shift parameter to move the set of equally spaced nodal points to an optimal position and raise the approximation order by one.

Technically, (8.1) is modified to

$$M_\ell^p = \sum_{m=0}^{M-1} \alpha_m^{p,s} (\frac{m}{2^s} + \tau_{p,s})^\ell, \quad \text{for } 0 \le \ell \le M, \tag{8.5}$$

for some prescribed integer $s \ge 0$. There are now $M+1$ variables to solve: $\alpha_m^{p,s}$ and $\tau_{p,s}$. To insure that the points $2^{-s}m + \tau_{p,s}$ lie in $[0, 2p-1]$, there are constraints for $\tau_{p,s}$ and s:

$$\frac{M-1}{2^s} < 2p - 1, \quad 0 \le \tau_{p,s} \le 2p - 1 - \frac{M-1}{2^s}.$$

Because of these constraints, there may not have a solution for each triplet (p, M, s). When there is a solution, the quadrature rule

$$\int f(x)\phi_{jk}(x)\, dx \approx \frac{1}{\sqrt{2^j}} \sum_{m=0}^{M-1} \alpha_m^{p,s} f(\frac{k + 2^{-s}m + \tau_{p,s}}{2^j})$$

is exact for polynomials of degree $\le M$, hence it is an order $M+1$ method.

The system of nonlinear equations (8.5) (in $\tau_{p,s}$) is highly ill-conditioned. Sweldens and Piessens provide an alternative by requiring that Chebyshev polynomials of degree $\le M$ are exactly integrated. Algorithms are provided in [36].

The multi-point methods can be derived in the same idea as (8.2) to the integrals of the forms (2.2)–(2.6). In this case we shall raise the levels to a same height. Then for each kind of combination of the common supports, we will need a set of parameters and nodal points. For instance the nodal points for $\int f(x)\phi^2(x)\, dx$ should be around $[0, 1]$ but those for $\int f(x)\phi(x)\phi(x-2)\, dx$ should be around $[2, 3]$.

When the integrals are evaluated on a finite interval, some ϕ_{jk} may "cross the boundary." The basic schemes of the multi-point rule do not address directly on this issue. However, it can be modified with the same spirit. For instance we first evaluate the integral

$$\int_0^\infty x^m \phi(x)\, dx,$$

including the case of $m = 0$. Then supply the results to compute the quadrature coefficients.

As the wavelets on closed intervals are getting widely applied, the boundary may no longer impose any difficulties.

9 More linear systems

Now we shall come back to the integrals with more integrands like those in (2.5) and (2.6). The basic forms are those with polynomials. For those we can consider the reduced forms like (3.4) and (3.5), with $f(x) = x^m$. One will find that the case with $m = 0$ can be solved as an eigenproblem as in Section 7 and the rest of them can be solved by linear systems of the same matrices like those in Section 6. Only now the matrices are more complicated.

Let us start with $m = 0$ and construct the associated matrix. Following the notations of Dahmen and Micchelli [15], suppose that $\phi_i(x)$, $0 \le i \le n$, are scaling functions (recall (3.1)) such that

$$\phi_i(x) = \sum_k c_i(k)\phi_i(2x-k).$$

For each $(k_1, \ldots, k_n) \in {}^n$, define

$$H(k_1, \ldots, k_n) = \int \phi_0(x)\phi_1(x-k_1)\ldots\phi_n(x-k_n)\,dx.$$

Then we have

$$H(k_1, \ldots, k_n) = \sum_{(\ell_1,\ldots,\ell_n)} a_{2k_1-\ell_1,\ldots,2k_n-\ell_n} H(\ell_1, \ldots, \ell_n),$$

where

$$a_{k_1,\ldots,k_n} = \frac{1}{2}\sum_\ell c_0(\ell)\Big(\prod_{i=1}^n c_i(\ell-k_i)\Big). \tag{9.1}$$

If $\phi_i(x)$ are either $\phi(x)$ or its derivative, we see that basicly the matrix is that constructed with $c^i(k)$ in (9.1) replaced by c_k. The derivatives involved will contribute a factor 2^s to the matrix. Hence for the case of $m = 0$ it is an eigenproblem, whose existence and uniqueness is guaranteed if ϕ is smooth enough. For higher m we will have a linear system of equations similar to (6.2), involving the same matrix here.

The multipoint scheme can be derived for general smooth functions $f(x)$. If it is too complicated to determine each case for all the combinations of ϕ_{jk}'s, it may be feasible now to first approximate $f(x)$ by a polynomial interpolation. Then approximate the integral by the interpolation polynomial. Theoretically the result is the same as a multi-point method.

In practice, one might question the efficiency of this complicated construction of matrices. Of course one can always choose the brutal force method: Evaluate $\phi(x)$ to fine enough resolution, multiply its pointwise values with $f(x)$ and apply the traditional trapezoidal or Simpson rules. As we mentioned in Section 7, Strang [34] has a method to determine the values of $\phi(x)$ at integers. It is an eigenproblem. Together with the techniques in Section 7 to normalize the eigenvectors formed by M_k and N_k, we can also determine the values of $\phi^{(s)}(x)$, if continuous, at integers. Once the integral values are settled, the scaling equation (1.1) can be applied to find the dyadic values (those at $k/2^d$).

Experiments show that the brutal force methods, although theoretically clumsy, may not be a problem for today's workstations in terms of the large data arrays and high number of operations. However, since the linear algebraic approaches are theoretically exact and the involved matrices are very small in today's scale, we have very reliable tools to solve the systems or eigenvectors. Therefore the brutal force methods are usually not as accurate as the linear algebraic methods. An example is that, with $p = 4$, using roughly 2^{11} data points, the trapezoidal rule got $\int \phi'(x)\phi(x)\,dx$ at the scale of 10^{-4}, while the answer should be 0.

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