COHEN'S CYCLES AND ORTHONORMAL SCALING FUNCTIONS*

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Abstract. There is a set of equivalence conditions for the orthonormality of the scaling functions. Among them, there is the Cohen's cycle condition. We will give more criteria for this condition: If the corresponding filter characteristic function has no zeros in certain closed intervals, then there are finitely many points to be checked for the validity of Cohen's cycle condition. However, these closed intervals cannot be made arbitrarily small.

Key words. orthogonal scaling function of compact support, Cohen's cycle

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1. Introduction. A function $\phi(x) \in L^2(\mathbb{R})$ that satisfies

$$\phi(x) = \sum_k c_k \phi(2x - k) \tag{1.1}$$

with $\int \phi \, dx = 1$ is called a *scaling function* with the *scaling coefficients* c_k. Suppose that $\phi(x)$ are compactly supported, and the coefficients c_k are nonzero only for $0 \le k \le N$ (for some integer N). Let

$$m_0(\xi) = \frac{1}{2} \sum_k c_k e^{ik\xi}$$

be the corresponding filter *characteristic function* for $\phi(x)$. Then $m_0(\xi)$ is a 2π-periodic trigonometric polynomial.

In order to build a family of wavelets of compact supports, Daubechies [3] started with the study of the characteristic function $m_0(\xi)$. Assume that $m_0(\xi)$ is a 2π-periodic trigonometric polynomial satisfying $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$, $m_0(0) = 1$, and having a factor

$$\left(\frac{1+e^{i\xi}}{2}\right)^p,$$

for some integer $p \ge 1$. It is summarized in Daubechies' "Ten Lectures" [4] that, with one of a few equivalence conditions (to be described below) the function $\phi(x)$ defined by the inverse Fourier transform of

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

is a compactly supported L^2-function that satisfies (1.1) and $\{\phi(x-k)\}$ is an orthonormal set. Here $\hat{\phi}(\xi)$ is the Fourier transform of $\phi(x)$, that is,

$$\hat{\phi}(\xi) = \int \phi(x) e^{-ix\xi} \, dx.$$

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When such a $\phi(x)$ is determined, the subspaces $V_j = \operatorname{span}\{\sqrt{2^j}\phi(2^j x - k) : k \in \mathbb{Z}\}$, for $j \in \mathbb{Z}$, form a multiresolution approximation of $L^2(\mathbb{R})$ and the functions $\sqrt{2^j}\psi(2^j x - k)$, for $j, k \in \mathbb{Z}$, constitute an orthonormal basis for $L^2(\mathbb{R})$. Here

$$\psi(x) = \sum_k (-1)^k c_{1-k}\phi(2x - k),$$

it is called a *wavelet.*

We now proceed to state the equivalent statements. The first statement is itself a set of equivalences:

$$\begin{aligned} & \{\phi(x-k)\} \text{ orthonormal} \\ \Longleftrightarrow\ & \{\sqrt{2^j}\psi(2^j x - k)\} \text{ orthonormal} \\ \Longleftrightarrow\ & \|\psi(x)\|_{L^2} = 1 \\ \Longleftrightarrow\ & \sum_k |\hat{\phi}(\xi + 2k\pi)|^2 = 1 \end{aligned} \tag{O}$$

Define $\alpha_\ell = \int \phi(x)\phi(x-\ell)\,dx$. Then, by (1.1) we have

$$\alpha_\ell = \frac{1}{2}\sum_m \left(\sum_k c_k c_{m-2\ell+k}\right)\alpha_m.$$

Define the square matrix $A = \frac{1}{2}(a_{\ell m})$ of proper dimension with $a_{\ell m} = \sum_k c_k c_{m-2\ell+k}$. We see that (α_ℓ) forms an eigenvector of A associated with eigenvalue 1. The following statement is by Lawton [6]:

$$\lambda = 1 \text{ is a nondegenerate eigenvalue of } A \tag{E}$$

Let $K \in \mathbb{R}$ be a compact set such that (1) the measure of K is 2π, (2) 0 is in the interior of K, and (3) for any $x \in [-\pi, \pi]$ there is an integer ℓ such that $x + 2\ell\pi \in K$. Note that K is actually a cut and shift-and-paste of the interval $[-\pi, \pi]$. The following statement is by Cohen [1]:

$$\text{There is such a } K \text{ that } |m_0(\xi)| \text{ has no zeros on } \frac{1}{2^m}K, \text{ for } m \geq 1 \tag{K}$$

Given an element $\xi \in [-\pi, \pi]$, let τ be the operator that $\tau\xi = 2\xi \pmod{2\pi}$. In the following, given a real number ξ, let $[\xi]$ be the (unique) number in $[-\pi, \pi)$ such that $[\xi] = \xi \pmod{2\pi}$. In other words, $\tau\xi = [2\xi]$. We now define the *τ-cycle of length k.*

Definition 1.1. *The set $\boldsymbol{\xi} = \{\xi_i \mid \xi_i \in [-\pi, \pi), 0 \leq i \leq k-1\}$ is a τ-cycle of length k if*

(1) $\xi_{i+1} = \tau\xi_i$;

(2) $\xi_i = \tau^k\xi_i$;

(3) $\xi_i \neq \xi_j$ *for any* $i \neq j$.

It is clear that if $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are two τ-cycles and their intersection is nonempty, then actually $\boldsymbol{\xi} = \boldsymbol{\eta}$. The only τ-cycle of length 1 is ([0]); we call it the trivial cycle. The *Cohen's cycle condition* is the following statement:

$$\begin{aligned} &\text{There is no nontrivial } \tau\text{-cycle } \boldsymbol{\xi} \text{ such that} \\ &m_0([\xi_k - \pi]) = 0 \text{ for all } \xi_k \in \boldsymbol{\xi} \end{aligned} \tag{C}$$

Finally, let T be the operator defined on 2π-periodic functions $f(\xi)$ such that

$$Tf(\xi) = \left|m_0(\frac{\xi}{2})\right|^2 f(\frac{\xi}{2}) + \left|m_0(\frac{\xi}{2}+\pi)\right|^2 f(\frac{\xi}{2}+\pi).$$

This operator is related to the matrix A by the observation that if $f(\xi) = \sum_k f_k e^{-ik\xi}$, then

$$Tf(\xi) = \frac{1}{2}\sum_{\ell}\sum_{m}\sum_{k} c_k c_{k-2\ell+m}\ f_m\ e^{-ik\xi}.$$

The last statement is:

The only trignometric polynomials that are invariant under T are the constants $\qquad (P)$

Then we have an equivalence relation among the previous statements:

$$(O) \Longrightarrow (K) \Longrightarrow (C) \Longrightarrow (P) \Longrightarrow (E) \Longrightarrow (O)$$

The proof can be found in the "Ten Lectures" [4], in which the original works by Cohen [1, 2] and Lawton [6] are cited. A simpler proof is discovered by Gröchenig [5].

Among the set of equivalence conditions, we will give a more detailed study of the Cohen's cycle condition (C). In the next section, we give some sufficient conditions for (O) based on the condition (C).

2. About the cycles. We list two simple facts about the τ-cycles, then we derive special cases about the Cohen's cycle condition. THEOREM 2.3 below, which covers a result by Mallat [7], was originally shown by Cohen [1] and also included in the "Ten Lectures" [4]. We include it here to make this article self-contained.

PROPOSITION 2.1. *Suppose* $([\xi_0], [\xi_1], \ldots, [\xi_{k-1}])$ *is a* τ*-cycle of length* k*. Then*

$$\xi_i = \frac{2m}{2^k - 1}\pi$$

for some $m \in \mathbb{Z}$*.*

Proof. By definition, $[2^k\xi_i] = [\xi_i]$ for all i. Thus $2^k\xi_i = \xi_i + 2m\pi$ for some $m \in \mathbb{Z}$ and the result follows. ∎

PROPOSITION 2.2. *Let*

$$\xi = \frac{2m}{2^k - 1}\pi \neq 0, \qquad \text{for some } m \in \mathbb{Z},$$

where k *is the smallest positive integer such that* ξ *has the above form. Then*

$$([\xi], [2\xi], \ldots, [2^{k-1}\xi])$$

is a τ*-cycle of length* k*.*

Proof. We check for the three conditions in . Condition (1) is easily satisfied. For condition (2), it suffices to check for $[2^k\xi] = [\xi]$. Indeed,

$$2^k\xi = 2^k\left(\frac{2m}{2^k-1}\pi\right) = \frac{2m}{2^k-1}\pi + 2m\pi = \xi + 2m\pi.$$

Now suppose $[2^i\xi] = [2^j\xi]$. Then $(2^i - 2^j)\xi = 2m_0\pi$ for some $m_0 \in \mathbb{Z}$. That is,

$$m_0 = \frac{2^i - 2^j}{2^k - 1}m.$$

Assume that $0 \le j < i \le k-1$, we see that 2^j is a factor of m_0. Thus

$$\xi = \frac{2 \cdot 2^{-j} m_0}{2^{i-j} - 1}\pi.$$

It contradicts to the assumption of k. Hence $i = j$. □

THEOREM 2.3. *Every non-trivial τ-cycle has an element in $[-\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \pi]$.*

Proof. Let $([\xi_0], [\xi_1], \ldots, [\xi_{k-1}])$ be a τ-cycle of length k with $k \ge 2$ such that there is no element in $[\frac{2}{3}\pi, \pi] \cup [-\pi, -\frac{2}{3}\pi]$. That is, for any i,

$$[\xi_i] \notin [\frac{2}{3}\pi, \pi] \cup [-\pi, -\frac{2}{3}\pi].$$

Since $k \ge 2$, we have

$$[\xi_{i+1}] \notin [-\frac{2}{3}\pi, 0] \cup [0, \frac{2}{3}\pi].$$

But i is arbitrary, therefore

$$[\xi_i] \notin [-\pi, \pi].$$

This contradicts to the definition of τ-cycles. □

By the above theorem, if we shift the τ-cycles by π, then they all have elements in $[-\pi/3, \pi/3]$. That means, if $m_0(\xi) \ne 0$ in the interval $[-\pi/3, \pi/3]$, then the cycle condition (C) is satisfied. Therefore $\{\phi(x-k)\}$ is an orthonormal set.

Let us note that, the only τ-cycle of length 1 is $([0])$, and the only τ-cycle of length 2 is $([-\frac{2}{3}\pi], [\frac{2}{3}\pi])$. In the next theorem, we will show that any τ-cycle of length ≥ 3 will have an element inside of $[-\pi, -\frac{4}{5}\pi] \cup [\frac{4}{5}\pi, \pi]$. Consequently, we have the following conclusion:

(2.1) If $m_0(\xi) \ne 0$ in the interval $[-\frac{\pi}{5}, \frac{\pi}{5}]$, and if $m_0(\frac{\pi}{3}) \ne 0$, then the cycle condition (C) is satisfied.

THEOREM 2.4. *Let $([\xi_0], [\xi_1], \ldots, [\xi_{k-1}])$ be a τ-cycle of length ≥ 3. Then there exists some $[\xi_i] \in [-\pi, -\frac{4}{5}\pi] \cup [\frac{4}{5}\pi, \pi]$.*

Proof. Let $([\xi_0], [\xi_1], \ldots, [\xi_{k-1}])$ be a τ-cycle of length k, where $k \ge 3$. Suppose that,

$$[\xi_i] \notin [-\pi, -\frac{4}{5}\pi] \cup [\frac{4}{5}\pi, \pi], \quad \forall i. \tag{2.2}$$

Then

$$[\xi_{i+1}] \notin [\frac{-2}{5}\pi, \frac{2}{5}\pi]. \tag{2.3}$$

Note that (2.2) can also be written as

$$[\xi_i] \ \notin [-\frac{6}{5}\pi, -\frac{4}{5}\pi] \cup [\frac{4}{5}\pi, \frac{6}{5}\pi] \pmod{2\pi},$$

therefore

$$[\xi_{i-1}] \ \notin [-\frac{3}{5}\pi, -\frac{2}{5}\pi] \cup [\frac{2}{5}\pi, \frac{3}{5}\pi]. \tag{2.4}$$

By (2.3) and (2.4), and since i is arbitrary, we have

$$[\xi_i] \ \notin [-\pi, -\frac{4}{5}\pi] \cup [-\frac{3}{5}\pi, \frac{3}{5}\pi] \cup [\frac{4}{5}\pi, \pi], \quad \forall i. \tag{2.5}$$

Now, the cycle has elements only inside of the intervals $(-\frac{4}{5}\pi, -\frac{3}{5}\pi)$ or $(\frac{3}{5}\pi, \frac{4}{5}\pi)$. That is,

$$[\xi_{i+1}] \in (-\frac{4}{5}\pi, -\frac{2}{5}\pi) \cup (\frac{2}{5}\pi, \frac{4}{5}\pi).$$

But, by (2.5),

$$[\xi_{i+1}] \notin [-\frac{3}{5}\pi, -\frac{2}{5}\pi] \cup [\frac{2}{5}\pi, \frac{3}{5}\pi].$$

Go back to $[\xi_i]$, we conclude that the cycle can only has elements inside of the intervals

$$(-\frac{1}{2}(\frac{3}{5} + \frac{4}{5})\pi \ , -\frac{3}{5}\pi) \quad \text{or} \quad (\frac{3}{5}\pi, \ \frac{1}{2}(\frac{3}{5} + \frac{4}{5})\pi). \tag{2.6}$$

In other words

$$[\xi_i] \ \notin [-\frac{4}{5}\pi, \ -\frac{1}{2}(\frac{3}{5} + \frac{4}{5})\pi] \cup [\frac{1}{2}(\frac{3}{5} + \frac{4}{5})\pi, \ \frac{4}{5}\pi], \quad \forall i. \tag{2.7}$$

Then again, by (2.6), we have

$$[\xi_{i+1}] \in (-\frac{4}{5}\pi, -\frac{3}{5}\pi) \cup (\frac{3}{5}\pi, \frac{4}{5}\pi).$$

But, by (2.7) and going back to $[\xi_i]$, actually the cycle can only appear in

$$(-\frac{1}{2}(\frac{3}{5} + \frac{4}{5})\pi, \ -\frac{1}{2}(\frac{3}{5} + \frac{1}{2}(\frac{3}{5} + \frac{4}{5}))\pi) \quad \text{or} \quad (\frac{1}{2}(\frac{3}{5} + \frac{1}{2}(\frac{3}{5} + \frac{4}{5}))\pi, \ \frac{1}{2}(\frac{3}{5} + \frac{4}{5})\pi),$$

which also implies that

$$\left.\begin{array}{l} [\xi_i] \notin [\frac{3}{5}\pi, \ \frac{1}{2}(\frac{3}{5} + \frac{1}{2}(\frac{3}{5} + \frac{4}{5}))\pi] \cup [\frac{1}{2}(\frac{3}{5} + \frac{4}{5})\pi, \ \frac{4}{5}\pi] \quad \text{or} \\ [\xi_i] \notin [-\frac{4}{5}\pi, \ -\frac{1}{2}(\frac{3}{5} + \frac{4}{5})\pi] \cup [-\frac{1}{2}(\frac{3}{5} + \frac{1}{2}(\frac{3}{5} + \frac{4}{5}))\pi, \ -\frac{3}{5}\pi] \end{array}\right. \quad \forall i.$$

Continue with the above cut-and-paste process we have the following result:

$$\left.\begin{aligned}[\xi_i] &\notin \bigcup_n \{[\frac{3}{5}\pi, a_n\pi] \cup [b_n\pi, \frac{4}{5}\pi]\} \\ [\xi_i] &\notin \bigcup_n \{[-\frac{4}{5}\pi, -b_n\pi] \cup [-a_n\pi, -\frac{3}{5}\pi]\}\end{aligned}\right. \quad \forall i$$

Here

$$a_0 = \frac{3}{5}, \quad b_0 = \frac{4}{5}, \quad \text{and} \quad b_n = \frac{a_{n-1} + b_{n-1}}{2}, \quad a_n = \frac{a_{n-1} + b_n}{2}.$$

We then solve the linear equation

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}.$$

A can be decomposed into BDB^{-1} with

$$B = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}.$$

We get

$$\lim_{n\to\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} (1/4)^n & 0 \\ 0 & 1^n \end{pmatrix} \begin{pmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}.$$

Hence we have $\lim a_n = \lim b_n = \frac{2}{3}$. That means $[\xi_i] \in \{[\frac{-2}{3}\pi], [\frac{2}{3}\pi]\}$. But it is a τ-cycle of length 2, contradicts to the assumption that $k \geq 3$. Therefore (2.2) is wrong and we have the theorem. □

By a similar cut-and-paste argument on the intervals (but with more labors), we have the following fact:

(2.8) If $m_0(\xi) \neq 0$ in the interval $[-\frac{3}{17}\pi, \frac{3}{17}\pi]$, and if $m_0(\frac{\pi}{3}) \neq 0$, $m_0(\frac{\pi}{5}) \neq 0$, then the cycle condition (C) is satisfied.

The proof has no new ideas, it is just a repetition of the techniques that are used in the previous proof. So we decide to omit it here and refer the interested readers to [8].

We observe a pattern from (2.1) and (2.8). That is, when $m_0(\xi) \neq 0$ in $[-\pi/3, \pi/3]$, the Cohen's cycle condition (C) is satisfied. When $m_0(\xi) \neq 0$ in a closed interval slightly smaller than $[-\pi/3, \pi/3]$, the situation is not getting much worse all of a sudden. Namely, we can still verify the cycle condition (C) by examining $m_0(\xi)$ at certain (finitely many) points.

It is natural to ask whether a sequence of propositions in the pattern of (2.1) and (2.8) can be established for arbitrarily small intervals? To be precise, we consider the intervals of the form

$$[-\alpha\pi, \alpha\pi], \text{ for } 0 < \alpha < 1. \tag{2.9}$$

Now we want to know, when $m_0(\xi) \neq 0$ inside of $[-\alpha\pi, \alpha\pi]$, whether or not there are finitely many points $\xi_1, \xi_2, \ldots, \xi_n$ such that if $m_0(\xi_i) \neq 0$, for $1 \leq i \leq n$, then the cycle condition (C) is satisfied? The answer is negative if α is too small. It is the subject of the last section.

3. The making of new cycles. We start with two more simple facts about the τ-cycles. Then we present a general algorithm for building a τ-cycle of length $2k$ from a τ-cycle of length k. Actually, we can build a τ-cycle of length $2mk$ from a τ-cycle of length k for any integer $m > 0$.

LEMMA 3.1. *Let $([\xi_0], [\xi_1], \ldots, [\xi_{k-1}])$ be a τ-cycle of length k. Then*

$$[\xi_i] = [\frac{\xi_{i+1}}{2} - \pi] \quad \text{or} \quad [\xi_i] = [\frac{\xi_{i+1}}{2}]$$

LEMMA 3.2. *Let $([\xi_0], [\xi_1], \ldots, [\xi_{k-1}])$ be a τ-cycle of length k, then $([-\xi_0], [-\xi_1], \ldots, [-\xi_{k-1}])$ is also a τ-cycle of length k.*

We omit the proofs for the above two simple facts.

THEOREM 3.1. *Let $([\xi_0], [\xi_1], \ldots, [\xi_{k-1}])$ be a τ-cycle of length k and*

$$2\gamma = \min\{|[\xi_i]| : 0 \le i \le k-1\}.$$

Without loss of generality (by LEMMA 3.2*), let $[\xi_0] = -2\gamma = \frac{2m_0}{2^k-1}\pi$. Then*

$$\{[2^i(\xi_{k-1} + \beta)] : 0 \le i \le 2k-1\}, \text{ where } \beta = \frac{2\gamma}{2^k+1}, \tag{3.1}$$

is a τ-cycle of length $2k$.

Proof. In order to prove that (3.1) is a τ-cycle of length $2k$, it suffices to show that

$$[2^i\xi_{k-1} + 2^i\beta] \ne [2^j\xi_{k-1} + 2^j\beta], \quad \forall\ 0 \le i \ne j \le k-1, \tag{3.2}$$

and

$$[2^k\xi_{k-1} + 2^k\beta] = [-(\xi_{k-1} + \beta)]. \tag{3.3}$$

To prove (3.2), suppose $[2^i\xi_{k-1} + 2^i\beta] = [2^j\xi_{k-1} + 2^j\beta]$ for some integers $0 \le i \ne j \le k-1$. It can be reduced to

$$[2^\ell\xi_{k-1} + 2^\ell\beta] = [\xi_{k-1} + \beta] \quad \Longrightarrow \quad (2^\ell - 1)\xi_{k-1} + (2^\ell - 1)\beta = 2m\pi,$$

for some integer m and $1 \le \ell \le k-1$. This means

$$\begin{aligned} 2m\pi &= (2^\ell - 1)\cdot(\frac{2^{k-1}\cdot 2m_0}{2^k-1}\pi - \frac{2m_0}{(2^k+1)(2^k-1)}\pi) \\ &= \frac{2^\ell-1}{2^k-1}\cdot\frac{2^k(2^k+1)m_0 - 2m_0}{2^k+1}\pi \\ &= \frac{2m_0(2^\ell-1)}{2^k-1}\cdot\frac{2^{k-1}(2^k+1)-1}{2^k+1}\pi. \end{aligned} \tag{3.4}$$

Since 2^k+1 and $2^{k-1}(2^k+1)-1$ are relative primes for any $k > 0$, $2^\ell - 1 < 2^k - 1$ and $|2m_0| < 2^k + 1$, so m cannot be an integer. This is a contradiction and (3.2) is proved.

Next we prove (3.3). Since $[\xi_0]$ is the absolute minimum, by LEMMA 3.1 and LEMMA 3.2, we have $[\xi_{k-1}] = [\frac{\xi_0}{2} - \pi] = [\pi - \gamma]$ or $[-\pi + \gamma]$. For our choice of $[\xi_0]$, $[\xi_{k-1}]$ should be positive. So we take $[\xi_{k-1}] = [\pi - \gamma]$. Then

$$\begin{aligned}
[2^k \xi_{k-1} + 2^k \beta] &= [\pi - \gamma + \frac{2^k \cdot 2\gamma}{2^k + 1}] \\
&= [\pi - \gamma + 2\gamma - \frac{2\gamma}{2^k + 1}] \\
&= [\pi + \gamma - \frac{2\gamma}{2^k + 1}] \\
&= [-\pi + \gamma - \frac{2\gamma}{2^k + 1}] \\
&= [-(\xi_{k-1} + \beta)].
\end{aligned}$$

Similar to (3.2), we have

$$[2^{k+i}\xi_{k-1} + 2^{k+i}\beta] \neq [2^{k+j}\xi_{k-1} + 2^{k+j}\beta], \quad \forall\, 0 \le i \neq j \le k-1.$$

And

$$[2^{2k}\xi_{k-1} + 2^{2k}\beta] = [-2^k(\xi_{k-1} - \beta)] = [\xi_{k-1} + \beta].$$

Therefore, (3.1) forms a τ-cycle of length $2k$. □

COROLLARY. *Let* $([\xi_0], [\xi_1], \ldots, [\xi_{k-1}])$ *be a* τ*-cycle of length* k, $2\gamma = \min\{|[\xi_i]| : 0 \le i \le k-1\}$, *and take* $[\xi_0] = -2\gamma$. *Then, for any integer* $m > 0$,

$$\{[2^i(\xi_{k-1} + \beta)] : 0 \le i \le 2mk - 1\}, \text{ where } \beta = \frac{2\gamma}{2^{mk} + 1},$$

is a τ*-cycle of length* $2mk$.

Here is an example. Let $[\xi_0] = -2\gamma = -\frac{2}{7}\pi$, then

$$\boldsymbol{\xi} = ([-\frac{2}{7}\pi], [-\frac{4}{7}\pi], [\frac{6}{7}\pi])$$

is a τ-cycle of length $k = 3$. Let $m > 0$ be any integer, define

$$[\beta] = \frac{2\gamma}{2^{mk} + 1} = \frac{2}{7(2^{3m} + 1)}\pi.$$

Then

$$\{[2^i(\frac{6}{7} + \frac{2}{7(2^{3m} + 1)})\pi] : 0 \le i \le 6m - 1\}$$

is a τ-cycle of length $6m$.

4. The lower bound for the shrinking intervals. In this section, we will construct infinitely many τ-cycles whose elements are always in the interval $(-\frac{6}{7}\pi, \frac{6}{7}\pi)$. We will see that the parameter α in (2.9) cannot be less than $1/7$.

Note that

$$([\frac{4}{7}\pi], [-\frac{6}{7}\pi], [\frac{2}{7}\pi]) \quad \text{and} \quad ([\frac{6}{7}\pi], [-\frac{2}{7}\pi], [-\frac{4}{7}\pi])$$

are two τ-cycles of length 3. We still let

$$\beta = \frac{2}{7(2^{3m}+1)}\pi.$$

Consider

$$\boldsymbol{\xi}_m := \{[2^i(\frac{4}{7}\pi + \beta)] : 0 \le i \le 6m-1\}.$$

Use (3.4), one can show that $2^i(\frac{4}{7}\pi + \beta)$ are all distinct for $0 \le i \le 3m-1$. Note that

$$[2^{3m}(\frac{4}{7}\pi+\beta)] = [\frac{4}{7}\pi + \frac{2\cdot 2^{3m}}{7(2^{3m}+1)}] = [\frac{4}{7}\pi + \frac{2}{7}\pi - \beta] = [\frac{6}{7}\pi - \beta].$$

Use (3.4) again, one can show that $2^i(\frac{6}{7}\pi - \beta)$ are all distinct for $0 \le i \le 3m-1$. But

$$[2^{3m}(\frac{6}{7}\pi-\beta)] = [\frac{6}{7}\pi - \frac{2\cdot 2^{3m}}{7(2^{3m}+1)}] = [\frac{6}{7}\pi - \frac{2}{7}\pi + \beta] = [\frac{4}{7}\pi + \beta].$$

We conclude that $\boldsymbol{\xi}_m$ is a τ-cycle of length $6m$, for any $m > 0$. The elements of $\boldsymbol{\xi}_m$ are:

$$\begin{array}{cccc} [(\frac{4}{7}+\frac{2\cdot 2^{3i}}{7(2^{3m}+1)})\pi] & [(-\frac{6}{7}+\frac{2\cdot 2^{3i+1}}{7(2^{3m}+1)})\pi] & [(\frac{2}{7}+\frac{2\cdot 2^{3i+2}}{7(2^{3m}+1)})\pi] & \\ & & & \text{for } 0 \le i \le m-1 \\ [(\frac{6}{7}-\frac{2\cdot 2^{3i}}{7(2^{3m}+1)})\pi] & [(-\frac{2}{7}-\frac{2\cdot 2^{3i+1}}{7(2^{3m}+1)})\pi] & [(-\frac{4}{7}-\frac{2\cdot 2^{3i+2}}{7(2^{3m}+1)})\pi] & \end{array}$$

Therefore, all elements for all $\boldsymbol{\xi}_m$ are inside of the interval $(-\frac{6}{7}\pi, \frac{6}{7}\pi)$. In other words, all of these τ-cycles of length $6m$ (for any integer $m > 0$) do not fall into the intervals $[-\pi, -\frac{6}{7}\pi]$ and $[\frac{6}{7}\pi, \pi]$.

The foregoing example shows that, after a shift of π, there are infinitely many τ-cycles whose elements do not fall into the interval $[-\frac{1}{7}\pi, \frac{1}{7}\pi]$. Then we have the following conclusion.

> If $m_0(\xi) \ne 0$ for $\xi \in [-\alpha\pi, \alpha\pi]$, where $\alpha \le \frac{1}{7}$, then there are infinitely many points to be checked for the validity of Cohen's cycle condition (C).

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