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Cyclic Reduction Method and Difference Wavelets *

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Abstract. The study of a family of direct methods for solving sparse banded linear systems motivated by multiresolution decomposition gives a new look of the classical cyclic reduction method. From this point of view we derive a family of difference wavelets associated with difference operators, which forms a generalized multiresolution analysis in a different sense of measure.

Key words. banded sparse matrix, cyclic reduction, multiresolution decomposition, wavelet

Subject classifications: AMS(MOS): 65F05, 65L12

1. Introduction. Fast algorithms for solving differential and integral equations are important for large-scale scientific computing. It usually brings down computational complexity for several orders of magnitude. This ability makes many difficult scientific and engineering problems become analyzable through computational methods.

Examples of fast algorithms are, for instance, multigrid methods, domain decomposition methods, multipole methods. Recently, a new direction of fast algorithms is being developed based on the wavelets theory.

Because of their similarities with Fourier transforms and finite elements, wavelets are studied as being a tool in the numerical solution of differential equations. Among the recent articles, we refer to the works by Bacry, Mallat and Papanicolaou [1], Beylkin, Coifman and Rokhlin [4], Glowinski, Lawton, Ravachol and Tenenbaum [6], Jaffard [7], Liandrat, Perrier and Tchamitchian [8], Maday, Perrier and Ravel [9] and Xu and Shann [10].

In [5], we took an approach that was motivated by the multiresolution analysis in the wavelets theory. Our approach was based on a decomposition process such that the size of the problem can be reduced and this process can be recursive. More precisely, our method is a fast direct method for solving banded sparse linear systems, which are originated from discretizing differential equations by finite difference or finite element methods. The method is of linear computational complexity.

This method was later found to be the same as the classical cyclic reduction method which appeared in late 60's [3]. Thus, our study [5] gave a new look of the cyclic reduction method from the point of view of multiresolution analysis. Further, we also found that the cyclic reductuion method induces a new sets of wavelets which we name them "difference wavelets" for their relation to the difference operators. These "wavelets" can be used to diagonalize difference operator D^p. They can be used for data compression. They can also be used as a hierarchical basis for constructing preconditioner for general difference equations.

2. Basic idea and generalized multiresolution analysis. Suppose we are given a normalized tridiagonal system

$$a_j u_{j-1} + u_j + c_j u_{j+1} = f_j, \quad \text{for } j = 1, \ldots, N, \tag{2.1}$$

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where N is the number of unknowns which is assumed to be $2^L - 1$ for some positive integer L. The basic idea is the multiresolutional decomposition. Suppose the numerical solution u_L of a differential equation is computed in a finite dimensional space V_L, we decompose

$$u_L = u_{L-1} + (u_L - u_{L-1}) = \cdots = u_0 + w_0 + \cdots + w_{L-1}.$$

Here u_ℓ is considered the resolution of u_L at level ℓ; while $w_\ell = u_{\ell+1} - u_\ell$ is the difference, or "detail," between the two resolutions. We want to design a sequence of spaces $V_0 \subset V_1 \subset \cdots \subset V_L$ such that

$$V_\ell = V_0 + W_0 + \cdots + W_{\ell-1}$$

with $u_\ell \in V_\ell$, $w_\ell \in W_\ell$. The decomposition shall satisfy the following criteria.

Suppose u_ℓ is solved by the linear system $A_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell$; here $A_L = A$. After one step of decomposition, we shall have the linear system

$$\begin{pmatrix} A_{\ell-1} & B_{\ell-1} \\ C_{\ell-1} & D_{\ell-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\ell-1} \\ \boldsymbol{w}_{\ell-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{\ell-1} \\ \boldsymbol{g}_{\ell-1} \end{pmatrix}.$$

We would like the decomposition to be done in spaces $V_{\ell-1}$ and $W_{\ell-1}$ such that, (1) $B_{\ell-1} = 0$ or $C_{\ell-1} = 0$; (2) $D_{\ell-1}^{-1}$ can easily be computed; and (3) $A_{\ell-1}$ keeps the same sparsity pattern as that of A_ℓ, such that the process can be applied recursively.

3. The cyclic reduction scheme. The cyclic reduction process will reduce system (2.1) to a similar system with roughly half size in every reduction step. This process is then applied recursively until we arrive a very small system which can easily be solved. From the point of view of multiresolutional analysis, we define the number of unknowns at level ℓ to be $N_\ell = 2^\ell - 1$, $1 \le \ell \le L$. The approximate solution space V_ℓ at level ℓ is defined to be $\mathbb{R}^{N_\ell}$, whose elements are denoted by $\boldsymbol{u}_\ell = (u_j^\ell)_{j=1}^{N_\ell}$. To illustrate the reduction process from level ℓ to level $\ell - 1$, let us assume that at level ℓ we have a system of equations

$$a_j^\ell u_{j-1}^\ell + u_j^\ell + c_j^\ell u_{j+1}^\ell = f_j^\ell, \quad \text{for } j = 1, \ldots, N_\ell,$$

We can eliminate the odd-numbered variables and reduce the system to the even-numbered variables by performing

$$-a_{2j}^\ell \times (2j-1) + (2j) - c_{2j}^\ell \times (2j+1),$$

then normalizing the result equations by $1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell$ and then relabeling the variables and coefficients. Finally we arrive the equation

$$a_j^{\ell-1} u_{j-1}^{\ell-1} + u_j^{\ell-1} + c_j^{\ell-1} u_{j+1}^{\ell-1} = f_j^{\ell-1}, \quad \text{for } j = 1, \ldots, N_{\ell-1}.$$

Here $u_j^{\ell-1} \equiv u_{2j}^\ell$. The new set of coefficients can be derived from those in the higher level:

$$\begin{aligned} a_j^{\ell-1} &= -\frac{a_{2j}^\ell a_{2j-1}^\ell}{1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell} \\ c_j^{\ell-1} &= -\frac{c_{2j}^\ell c_{2j+1}^\ell}{1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell} \\ f_j^{\ell-1} &= \frac{f_{2j}^\ell - a_{2j}^\ell f_{2j-1}^\ell - c_{2j}^\ell f_{2j+1}^\ell}{1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell} \end{aligned}$$

Thus, at level ℓ if we define the change-of-variable

$$
\begin{aligned}
u_j^{\ell-1} &= u_{2j}^{\ell} \\
w_j^{\ell-1} &= a_{2j+1}^{\ell} u_{2j}^{\ell} + u_{2j+1}^{\ell} + c_{2j+1}^{\ell} u_{2j+2}^{\ell}
\end{aligned}
\tag{3.1}
$$

in the domain of A_ℓ and the change-of-variable

$$
\begin{aligned}
f_j^{\ell-1} &= \frac{f_{2j}^{\ell} - a_{2j}^{\ell} f_{2j-1}^{\ell} - c_{2j}^{\ell} f_{2j+1}^{\ell}}{1 - a_{2j}^{\ell} c_{2j-1}^{\ell} - a_{2j+1}^{\ell} c_{2j}^{\ell}} \\
g_j^{\ell-1} &= f_{2j+1}^{\ell}
\end{aligned}
$$

in the range of A_ℓ, then the system $A_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell$ is reduced to

$$
\begin{pmatrix} A_{\ell-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\ell-1} \\ \boldsymbol{w}_{\ell-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{\ell-1} \\ \boldsymbol{g}_{\ell-1} \end{pmatrix}.
$$

Thus, the change-of-variable

$$
\begin{aligned}
u^L &\longleftrightarrow (u^1, w^1, w^2, \cdots, w^{L-1}) \\
f^L &\longleftrightarrow (f^1, g^1, g^2, \cdots, g^{L-1})
\end{aligned}
$$

diagonalizes the system. From $(u^1, w^1, w^2, \cdots, w^{L-1})$ we can find u^L recursively by

$$
\begin{aligned}
u_{2j}^{\ell} &= u_j^{\ell-1} \\
u_{2j+1}^{\ell} &= w_j^{\ell-1} - a_{2j+1}^{\ell} u_j^{\ell-1} - c_{2j+1}^{\ell} u_{j+1}^{\ell-1}.
\end{aligned}
$$

The cyclic reduction method is stable for diagonally dominant matrix. Some other sufficent conditions for stability of the cyclic reduction method were reported in [2,5].

4. Relations to hierarchical bases and the difference wavelets. An interesting case for the cyclic reduction method is the case when the coefficients (a_j^ℓ, c_j^ℓ) are constants. It is easy to check that this only occurs when $(a, 1, c) = (-1, 1, 0)$, or $(0, 1, -1)$, or $(-1/2, 1, -1/2)$, which correspond to the backward, forward and central difference operators, denoted by D_-, D_+ and D^2, respectively. When applying the cyclic reduction method to matrices with wider bandwidth, we find that only those matrices generated from the difference operators D^p, $p \in \mathbb{Z}$ are invariant. Thus, the cyclic reduction method provides a change-of-variable to diagonalize these kinds of matrices. Now from this change-of-variable, we show that for each $p \in \mathbb{Z}$ there exists a generalized multiresolution analysis and a hierarchical basis under which the difference operator D^p is diagonal. In the explanation of the idea below, we shall take D^2 as an example.

First, let us briefly review the concept of multiresolution analysis. The notion of multiresolution analysis was introduced by Mallat to provide a framework for wavelet theory. According to him, multiresolution analysis consists of a sequence of successive approximation spaces $\cdots \subset V_{\ell-1} \subset V_\ell \subset \cdots \subset L^2(\mathbb{R})$ such that (i) $\overline{\bigcup V_\ell} = L^2$ and $\bigcap V_\ell = \phi$, (ii) $u(\cdot) \in V_\ell \leftrightarrow u(2^{-\ell}\cdot) \in V_0$, (iii) $u(\cdot) \in V_0 \leftrightarrow u(\cdot - j) \in V_0$, $\forall j \in \mathbb{Z}$, and (iv) there exists $\phi \in V_0$ such that $\{\phi(\cdot - j) \mid j \in \mathbb{Z}\}$ is an orthonormal basis in V_0. Notice that $\{\phi_j^\ell \equiv \sqrt{2^\ell}\phi(2^\ell - j) \mid j \in \mathbb{Z}\}$ forms an orthonormal basis in V_ℓ. ϕ is called the scaling function. Let $W_{\ell-1}$ be the orthogonal complement of $V_{\ell-1}$ in V_ℓ. Then $L^2 = \overline{\bigoplus W_\ell}$. Moreover, there exists a function $\psi \in W_0$ (called mother wavelet) such that the functions $\{\psi_j^\ell \equiv \sqrt{2^\ell}\psi(2^\ell - j) \mid j, \ell \in \mathbb{Z}\}$ forms an orthonormal basis in $L^2(\mathbb{R})$.

To construct a scaling function ϕ and a multiresolution analysis associated to the difference operator D^2, we first require that ϕ has the property: $\phi(0) = 1$ and $\phi(j) = 0$, otherwise. Suppose at level ℓ, our grids are $\mathcal{T}_\ell = \{x_j^\ell\}$, $x_j^\ell = j\, h_\ell$ and $h_\ell = 2^{-\ell}$. Then

$$u(x_k^\ell) = \sum_j u_j^\ell \phi_j^\ell(x_k^\ell),$$

where $u_j^\ell = u(x_j^\ell$ and $\phi_j^\ell(\cdot) \equiv \phi(2^\ell \cdot - j)$. Let V_ℓ =span$\{\phi_j^\ell \mid j \in \mathbb{Z}\}$. It is easy to see that $V_{\ell-1} \subset V_\ell$. We shall decompose $V_\ell = V_{\ell-1} \oplus W_{\ell-1}$ in such a way that there exists a function $\psi \in W_0$ such that

$$\sum_j u_j^\ell \phi_j^\ell = \sum_j u_j^{\ell-1}\phi_j^{\ell-1} + w_j^{\ell-1}\psi_j^{\ell-1}, \tag{4.1}$$

where the relation between u_j^ℓ and $(u_j^{\ell-1}, w_j^{\ell-1})$ are given by the change-of-variable (3.1). This change-of-variable induces a relation between $\phi^{\ell+1}$ and (ϕ^ℓ, ψ^ℓ). Because diag$(-1/2, 1, -1/2)$ is invariant under the cyclic reduction process, these relations (4.1) are identical for each level. In particular, at level 0 we have

$$\begin{aligned}
\phi(2x) &= \phi(x) + \frac{1}{2}\psi(x) + \frac{1}{2}\psi(x-1)\\
\phi(2x+1) &= \psi(x).
\end{aligned}$$

From this we deduce that $\phi(x)$ is the hat function (i.e. ϕ is piecewise linear, $\phi(0) = 1$ and $\phi(j) = 0$ otherwise). Notice that $\{\phi(\cdot - j) \mid j \in Z\}$ no longer forms a Riesz basis in $L^2(\mathbb{R})$. However, if we define a measure $d\mu_\ell$ in V_ℓ by $d\mu_\ell = \sum_j \delta_j$ (δ is the delta measure) and the inner product in V_ℓ by $(u, v)_\ell = \int u(x)v(x)d\mu_\ell$, then $\{\phi_j^\ell \mid j \in \mathbb{Z}\}$ forms a Riesz basis in V_ℓ with this inner product. Thus, $(V_\ell, d\mu_\ell)$ forms a generalized multiresolution analysis. We also notice that the functions ψ_j^ℓ diagonalizes the difference operator D^2 in the sense that $\int D\psi_i^k D\psi_j^\ell\, dx = 2\delta_{k\ell}\delta_{ij}$. We shall call the function ψ the *difference wavelet* of order 2.

The above difference wavelets of order 2 can be extended to any order p. Figure 1 shows the scaling functions and the difference wavelets of order 3 and 4. Notice that for smooth function u, as expanded in the difference wavelets of order p, its coefficients $w_j^\ell \equiv D^p u(x_j^\ell)$ are of $O(h_\ell^p)$. We also notice that the transformation and its inverse transformation between u^ℓ and $(u^{\ell-1}, w^{\ell-1})$ are of linear computational complexity. Thus, these difference wavelets can be used for data compression in signal processing. Further, since the difference wavelets diagonalize the difference operator, they can also be used as a hierachical basis for constructing a preconditioner for solving more general differtential equations.

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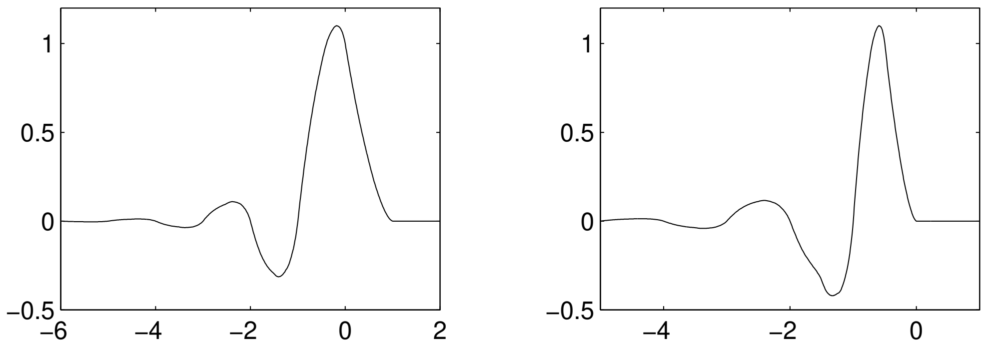
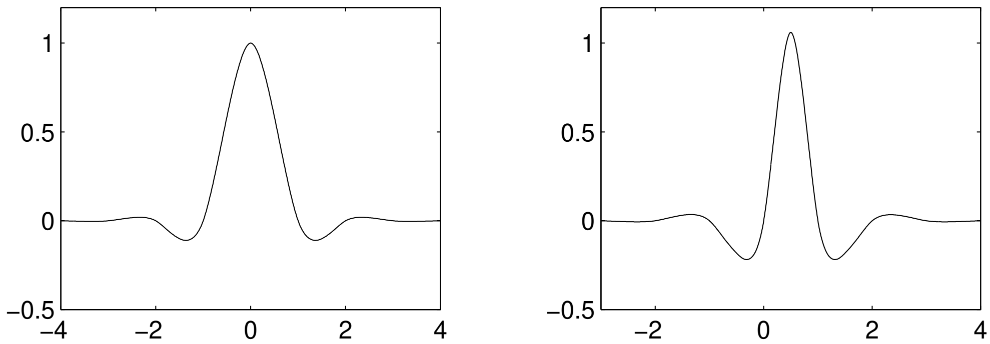


Difference scaling function (left) and wavelet (right) of order 3.



Difference scaling function (left) and wavelet (right) of order 4.

Figure 1.

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