A SCHEME TO ELEVATE THE DEGREE AND REGULARITY OF BIORTHOGONAL WAVELETS*

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Abstract. We will demonstrate a scheme that can elevate orthonormal or biorthogonal wavelets to a new pair of biorthogonal wavelet. The approximation degree, that is, the number of vanishing moments, of the elevated biorthogonal wavelet will be higher than the original.

Starting with B-splines and cooperating with the *lifting scheme* (Sweldens), our elevation scheme can reproduce the biorthogonal wavelets of compact support (Cohen, Daubechies, Feauveau). There is a simple algorithm to elevate from old filter coefficients to the new filter coefficients.

Starting with orthonormal wavelets, the elevated scaling functions are differentiable and the derivatives are nearly orthonormal to their translations. That is, the inner products of the translated derivatives always have the values $\{-1, 2, -1\}$. This feature resembles the *difference wavelets* (Chern, Shann) which may be valuable for solving differential equations.

Key words. biorthogonal wavelets, regularity, convolution, approximation degree, stiffness matrix, condition number.

1. Motivation. Since the introduction of *wavelets* ten years ago, there are intensive research activities in this area, in both theoretical and applied aspects. For more introductory details, we would like to refer the readers to [5, 10]. Among the applications, there are the numerical solution for integral equations and differential equations. The wavelets based numerical methods for the solution of integral equations are reasonably successful—at least, for those with kernels which do not oscillate too much. For this, people basically follow the work of [1]. On the other hand, the wavelets based numerical methods for the solution of differential equations are not as successful as those for integral equations. There are different points of views on why it might be difficult and how it might be resolved. Among them, there are groups of researchers who are looking for the "right" wavelets which are "made" for the differential equations.

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But what make wavelets attractive to the numerical solution of differential equations? We may not have the full answer (yet), but there are some plausible reasonings. The wavelets have simple structures, they have good approximation ability, they have compact or "short" supports, they can localize functions in both time and frequency domain, and there is a fast algorithm to change bases between high and low resolutions. In the long run, we wish to develop wavelets based numerical methods for the solution of differential equations with mixed large and small scales of motion.

Ideally, we wish to have wavelets which are in the appropriate function spaces, for instance the Sobolev space $H^1(\Omega)$, and satisfy the boundary conditions. We wish them to have compact or "short" supports. And when the differential equations are to be discretized into linear systems, we wish the wavelets to produce both *sparse* and *well-conditioned* matrices. There are several works published in this direction. For instance, Jaffard [6], Xu and Shann [12], Jawerth and Sweldens [7], Liandrat and Tchamitchian [8], Chern and Shann [2], Ponenti and Liandrat [9]. The work [2] is more related to the difference method, while the others are based on the Galerkin method to discretize the differential equations.

When we follow the framework of the Galerkin method to discretize a differential equation, it is the stiffness matrix that dominate the condition number of the resulted linear system. And, empirically, it is unlikely that the stiffness matrix will be both sparse and well-conditioned. In practice, people look for sparse pre-conditioners. One way of achieving this goal is to have wavelets whose derivatives are orthonormal and the associated scaling functions are compactly supported.

At the end of [12], Xu and Shann suggested to construct basis functions by

$$\int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt. \tag{1.1}$$

We follow this hint and found it is a special case as in [7]. But we can go further to have general results. It turns out that (1.1) is a general way to obtain from a wavelet to another one, with better properties but also with some limitations. We call this an *elevation* of wavelets. It is also very interesting to realize that, the elevation scheme and the *lifting* scheme [11] of Sweldens can work together to produce all of the compactly supported biorthogonal wavelets of Cohen, Daubechies and Feauveau [4]. And this procedure also gives us better understanding of these useful wavelets.

We will quickly review the fundamentals about wavelets in the next section. Then we will present the theoretical main results and leave the numerical experiments to a future work.

2. Fundamentals. Let $\phi(x)$ be a (two-scale) *scaling function* which satisfies a finite (two-scale) equation

$$\phi(x) = \sum c_k \phi(2x - k). \tag{2.1}$$

There are finitely many non-zero *scaling coefficients* c_k, say $c_0, \cdots, c_N$. Since the only interesting case is that ϕ is a low-pass filter, so we require $\hat{\phi}(0) \neq 0$, where $\hat{\phi}(\omega)$ is the Fourier transform of $\phi(x)$. Throughout this article we only consider the normalized case that $\hat{\phi}(0) = 1$.

THEOREM 2.1. *Let $c_0, \ldots, c_N \in \mathbb{R}$ with $\sum c_n = 2$. Then there is a unique compactly supported distribution ϕ satisfies* (2.1) *with $\hat{\phi}(0) = 1$. The support of ϕ is contained in* $[0, N]$.

In this article, when we say that $\phi(x)$ is a scaling function, we mean that $\phi(x)$ satisfies a scaling relation and the linear combinations of $\phi(x-k)$ can express polynomials up to a certain degree. The power of approximation depends on the scaling coefficients. We have the following proposition.

PROPOSITION 2.2. *Let $\phi(x)$ be defined as in* (2.1). *If the scaling coefficients c_k satisfy*

$$\sum (-1)^k k^m c_k = 0,$$

for integer $0 \leq m \leq p$, and the scaling function satisfies

$$\sum \phi(x-k) = 1 \quad \forall x \in \mathbb{R},$$

then $\sum k^m \phi(x-k)$ is a polynomial of degree m.

Define the *characteristic* function for $\phi(x)$ to be

$$m_0^{\phi}(\omega) = \frac{1}{2} \sum c_k e^{-ik\omega}.$$

$\phi(x)$ is said to be an orthonormal scaling function if $\{\phi(x-k)\}$ forms an orthonormal set. If $\phi(x)$ is an orthonormal scaling function, we have

$$\sum |\hat{\phi}(\omega + 2k\pi)|^2 = \frac{1}{2\pi} \quad \text{a.e.}$$

and

$$|m_0^\phi(\omega)|^2 + |m_0^\phi(\omega+\pi)|^2 = 1. \quad \text{a.e.}$$

Let V_j be the subspaces of $L^2(\mathbb{R})$ spanned by $\{\sqrt{2^j}\phi(2^j x - k)\}_k$. If there are constants $0 < A \le B < \infty$ such that

$$A \le \sum |\hat{\phi}(\omega + 2k\pi)|^2 \le B, \quad \text{a. e.}$$

then $\{\phi(x-k)\}$ forms a Riesz basis of V_0. Note that if $\phi(x)$ is an orthonormal scaling function, then $\{\phi(x-k)\}$ forms an orthonormal basis of V_0, which is a special case of a Riesz basis. If $\{\phi(x-k)\}$ is a Riesz basis, then

$$\cap V_j = \{0\} \text{ and } \overline{\cup V_j} = L^2(\mathbb{R}).$$

By (2.1), $V_j \subset V_{j+1}$. Let W_j be the orthogonal complement of V_j in V_{j+1}. Then $\oplus W_j = L^2$. If there is a function $\psi(x) \in W_0$ such that W_j is actually spanned by $\{\sqrt{2^j}\psi(2^j x - k)\}_k$, then $\psi(x)$ is called the *wavelet* associated with the scaling function $\phi(x)$. It turns out that $\psi(x)$ can be chosen such that its Fourier transform is

$$\hat{\psi}(\omega) = e^{i\frac{\omega}{2}}\overline{m_0^\phi(\frac{\omega}{2}+\pi)}\hat{\phi}(\frac{\omega}{2}) := m_1^\phi(\frac{\omega}{2})\,\hat{\phi}(\frac{\omega}{2}). \tag{2.2}$$

If $\phi(x)$ is an orthonormal scaling function, then $\{\psi(x-k)\}$ constructed from (2.2) is also an orthonormal set. In this case, we say $\psi(x)$ is an orthonormal wavelet, and thus $\{\sqrt{2^j}\psi(2^j x - k)\}_{j,k}$ is an orthonormal basis of $L^2(\mathbb{R})$. Note that $\psi(x) \in V_1$; indeed

$$\psi(x) = \sum (-1)^k c_{1-k}\phi(2x-k).$$

If $\phi(x)$ is not an orthonormal scaling function, there are mainly two ways to construct (more generalized) the associated wavelets. One is the semi-orthogonal approach by Chui and Wang [3], and another the biorthogonal approach by Cohen, Daubechies and Feauveau [4]. Both of them use the B-spline as the (non-orthonormal) scaling function. We will follow the later approach.

Let $\phi(x)$ be a scaling function which generates a sequence of subspaces V_j, and $\tilde{\phi}(x)$ be another scaling function which generates another sequence of subspaces $\tilde{V}_j$, $\{\phi(x-k)\}$ is a Riesz bases for V_0 and $\{\tilde{\phi}(x-k)\}$ is a Riesz bases for $\tilde{V}_0$. We say $\tilde{\phi}(x)$ and $\phi(x)$ are *biorthogonal* to each other if

$$\langle \tilde{\phi}(\cdot - k), \phi(\cdot - \ell)\rangle = \delta_{k\ell},$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product and $\delta_{k\ell}$ is the Dirac delta notation.

Let $\tilde{c}_0, \cdots, \tilde{c}_{\tilde{N}}$ be the scaling coefficients for $\tilde{\phi}(x)$, we can define the characteristic functions $m_0^{\tilde{\phi}}(\omega)$ and $m_1^{\tilde{\phi}}(\omega)$ analogously. If we can construct these functions such that

$$m_0^{\tilde{\phi}}(\omega)\,\overline{m_0^{\phi}(\omega)} + m_0^{\tilde{\phi}}(\omega+\pi)\,\overline{m_0^{\phi}(\omega+\pi)} = 1 \quad \text{a.e.} \tag{2.3}$$

and

$$\widetilde{M}(\omega)\,\overline{M^t(\omega)} = 1 \quad \text{a.e.} \tag{2.4}$$

then we will have a *biorthogonal system* $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$. Here

$$M(\omega) = \begin{pmatrix} m_0^{\phi}(\omega) & m_0^{\phi}(\omega+\pi) \\ m_1^{\phi}(\omega) & m_1^{\phi}(\omega+\pi) \end{pmatrix},$$

and $\widetilde{M}(\omega)$ is similarly defined.

THEOREM 2.3. *Let ϕ, $\tilde{\phi}$, $m_0^{\phi}(\omega)$ and $m_0^{\tilde{\phi}}(\omega)$ be defined as above. Then $m_1^{\phi}(\omega)$ and $m_1^{\tilde{\phi}}(\omega)$ satisfy* (2.4) *if and only if they are of the form*

$$m_1^{\tilde{\phi}}(\omega) = e^{-i\omega}\,\overline{m_0^{\phi}(\omega+\pi)}\,s(2\omega) \text{ and } m_1^{\phi}(\omega) = e^{-i\omega}\,\overline{m_0^{\tilde{\phi}}(\omega+\pi)}\,s(2\omega),$$

where $s(\omega)$ is some trigonometric polynomial bounded away from 0.

The simplest choice is that $s(\omega) = 1$. In this case,

$$\psi(x) = \sum (-1)^k \tilde{c}_{1-k}\phi(2x-k) \text{ and } \tilde{\psi}(x) = \sum (-1)^k c_{1-k}\tilde{\phi}(2x-k).$$

3. Main Results. Now let $\phi(x)$ be a compactly supported orthonormal scaling function which satisfies a scaling equation (2.1) with finitely many nonzero scaling coefficients $c_0, c_1, \ldots, c_{2p-1}$. The constant p is designated as the *approximation order* of the wavelet $\psi(x)$. That is, $\{\phi(x-n)\}_n$ can span polynomials of degree up to $p-1$ in any finite interval. Let $\psi(x)$ be the compact support orthonormal wavelet associated with $\phi(x)$, then

$$\langle x^m, \psi(x)\rangle = 0, \quad 0 \le m \le p-1.$$

We also review that, by the orthonormality, the scaling coefficients $\{c_n\}_{n=0}^{2p-1}$ must satisfy the following conditions.

$$(3.1)\qquad \begin{aligned} &(\mathrm{S}) \sum_{n=0}^{2p-1} c_n = 2, \\ &(\mathrm{A}) \sum_{n=0}^{2p-1} (-1)^n n^m c_n = 0 \text{ where } m = 0, \ldots, p-1 \quad (0^0 := 1), \\ &(\mathrm{O}) \sum_{n=0}^{2p-1} c_n c_{n-2m} = \delta_{0m}, \text{ for all } m \in \mathbb{Z}. \end{aligned}$$

In this case, $\mathrm{supp}\phi(x) = [0, 2p-1]$ and $d_n = (-1)^n c_{1-n}$, hence $\mathrm{supp}\psi(x) = [1-p, p]$.

Following up the idea in [12], define

$$(3.2)\qquad \Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt.$$

We call $\Phi(x)$ the *elevation* of $\phi(x)$. We will show that $\Phi(x)$ is still a scaling function that generates a multiresolution analysis. Therefore, there is a wavelet $\Psi(x)$ associated with $\Phi(x)$. Neither $\Phi(x)$ nor $\Psi(x)$ will be orthonormal. Furthermore, the derivatives

$$\Psi'(x-n) = \psi(x-n)$$

are orthogonal to each other. However, we will construct a pair of functions $\{\tilde{\Phi}(x), \tilde{\Psi}(x)\}$ such that $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$ forms a system of biorthogonal wavelets. With the biorthogonal structure, we will have fast algorithms to change the bases between V_{k+1} and $V_k + W_k$.

By the definition of $\Phi(x)$, we have

$$\begin{aligned} \Phi(x) &= \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt \\ &= \int_{-\infty}^{x} [\sum_{n=0}^{2p-1} c_n \phi(2t-n)] - [\sum_{n=0}^{2p-1} c_n \phi(2t-n-2)]\, dt \\ &= \int_{-\infty}^{x} \sum_{n=0}^{2p} \frac{(c_{n-1} + c_n)}{2} [\phi(2t-n-2) - \phi(2t-n-1)]\, dt \\ &:= \sum_{n=0}^{2p} h_n \Phi(2x-n). \end{aligned}$$

Here

$$(3.3)\qquad h_n = \frac{(c_{n-1} + c_n)}{2},$$

note that $c_{-1} = c_{2p} = 0$. Consequently, $\Phi(x)$ satisfies a finite two-scale relation with the scaling coefficients $\{h_n\}_{n=0}^{2p}$. Hence $\text{supp}\Phi(x) = [0, 2p]$. Furthermore, it is easy to see that

$$\sum_{n=0}^{2p} h_n = \sum_{n=0}^{2p-1} c_n = 2, \tag{3.4}$$

and

$$\sum_{n=0}^{2p} (-1)^n h_n = 0.$$

Proposition 3.1. *If $\phi(x)$ is a scaling function that satisfies* (2.1) *and $\Phi(x)$ is as defined in* (3.2). *Then*

$$\Phi(x) = \chi_{[0,1]} * \phi(x). \tag{3.5}$$

Furthermore, $\Phi(x)$ is a scaling function with scaling coefficients

$$(h_n) = (\frac{1}{2}, \frac{1}{2}) * (c_n).$$

Note that $\widehat{\chi_{[0,1]}}(\omega)$ is continuous at $\omega = 0$. Since $\hat{\phi}(\omega)$ is also a continuous function, so $\Phi(\omega)$ is continuous. Thus, we have $\hat{\Phi}(0) = 1$ and $\hat{\Phi}(2n\pi) = 0$ for $n \neq 0$.

Among the properties of c_n listed in (3.1), (S) follows from (2.1), (A) is necessary and sufficient for the degree of approximation, and (O) is necessary for the orthonormality. In (3.4), we have seen that h_n satisfies (S). Now we will show that h_n will satisfy (A) with one degree higher than that of c_n, but h_n will not satisfy (O).

Proposition 3.2. *If $\phi(x)$ is a scaling function that satisfies* (2.1) *and $\{c_n\}$ satisfies the condition* (A) *in* (3.1), *and if $\Phi(x)$ is as defined in* (3.2). *Then the scaling coefficients $\{h_n\}$ of $\Phi(x)$ satisfy*

$$\sum_{n=0}^{2p} (-1)^n n^m h_n = 0, \ \text{where } m = 0, \ldots, p \ (\text{with } 0^0 := 1). \tag{3.6}$$

The above proposition means that polynomials up to degree p can be a linear combination of the translations of $\Phi(x)$, if provided $\sum \Phi(x-n) \equiv 1$. It is well known that the $\{\phi(x-n)\}_n$ can express the constant functions, namely $\sum \phi(x-n) \equiv 1$. Although $\{\Phi(x-n)\}_n$ is not an orthonormal set (we will show that in the next section), Φ still preserves this property of ϕ such that $\sum \Phi(x-n) \equiv 1$.

Proposition 3.3. *If $\phi(x)$ is a scaling function such that*

$$\sum_n \phi(x-n) = 1, \quad \forall x \in \mathbb{R},$$

and $\Phi(x)$ is as defined in (3.2). *Then*

$$\sum_n \Phi(x-n) = 1, \quad \forall x \in \mathbb{R}.$$

By the fact that $\{\phi(x-n)\}$ can span the polynomials up to degree $p-1$ and the above proposition, we can prove that $\{\Phi(x-n)\}$ can span the polynomials up to degree p. Specifically,

$$\sum_n n^m \Phi(x-n)$$

is a polynomial of degree m, for $0 \leq m \leq p \quad (0^0 := 1)$.

Let $V_0 = \mathrm{span}\{\Phi(x-n)\}$, we would like to check if $\{\Phi(x-n)\}$ forms a Riesz basis of V_0. It amounts to show that there are constants $0 < A \leq B$ with

$$A \leq \sum_{n=-\infty}^{\infty} |\hat{\Phi}(\omega + 2\pi n)|^2 \leq B, \quad \forall \omega \in \mathbb{R}.$$

This is a well-known property, see, for example, [Chui]. If $\{\Phi(x-n)\}$ forms a Riesz basis of V_0 and by $\hat{\Phi}(0) = 1$, then $\Phi(x)$ generates a multiresolution analysis of $L^2(\mathbb{R})$.

Theorem 3.4. *If ϕ is a scaling function that generates a multiresolution analysis and $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the function $\Phi = \phi * \chi_{[0,1]}$ also generates a multiresolution analysis.*

We have seen that $\Phi(x)$ generates a multiresolution analysis. At this point, we do not know whether $\Phi(x-n)$ are orthogonal or not. However, by the numerical tests on several basic cases, we believe they are not orthogonal. Therefore, we consider the general setting of the biorthogonal system, see (2.3). We will derive the associated wavelet $\Psi(x)$ from the dual of $\Phi(x)$. The new biorthogonal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$ must satisfy (2.4). Because

$$m_0^{\Phi}(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right) m_0^{\phi}(\omega), \tag{3.7}$$

we chose

$$m_0^{\tilde{\Phi}}(\omega) = \begin{cases} \dfrac{2e^{-i\omega}}{1+e^{-i\omega}} m_0^{\tilde{\phi}}(\omega) & \text{if } \omega \neq 2\pi n, \\ 1 & \text{otherwise}, \end{cases} \tag{3.8}$$

and

$$m_1^{\tilde{\Phi}}(\omega) = e^{-i\frac{\omega}{2}} \overline{m_0^{\Phi}(\omega+\pi)}, \qquad m_1^{\Phi}(\omega) = e^{-i\frac{\omega}{2}} \overline{m_0^{\tilde{\Phi}}(\omega+\pi)}.$$

Then, see Section 4 of Chapter 2, we can construct the biorthogonal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$. For this chosen, we will see that (1) the relation between $\tilde{\Phi}$ and $\tilde{\phi}$ is "reciprocal" to the relation between Φ and ϕ; (2) $\{\Psi'(x-n)\}$ are orthogonal to each other and to $\{\Phi'(x-n)\}$; (3) $\{\Phi(x-n)\}$ are *not* orthogonal.

Proposition 3.5. *Let ϕ be an orthonormal scaling function and let Φ be defined by (3.5). If $\tilde{\Phi}$ is defined by*

$$\chi_{[-1,0]} * \tilde{\Phi} = \phi. \tag{3.9}$$

Then $\tilde{\Phi}$ is a biorthogonal dual of Φ.

In practice, the function $m_0^{\tilde{\phi}}(\omega)$ will have a factor of $(1+e^{-i\omega})/2$. Therefore, in practice, $m_0^{\tilde{\Phi}}(\omega)$ has finitely many coefficients; in other words, $\tilde{\Phi}(x)$ is compactly supported. However, we can see from (3.9) that the regularity of $\tilde{\Phi}(x)$ will be less than that of $\tilde{\phi}(x)$. We have seen this phenomenon in the biorthogonal wavelets in [4].

We have no regard for whether the pair $\{\tilde{\Phi}, \tilde{\Psi}\}$ exist in $L^2(\mathbb{R})$. As long as the scaling coefficients $(\tilde{c}_n)$ is a finite sequence, $\tilde{\Phi}$ and $\tilde{\Psi}$ are distributions at least.

We know that $\Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt$ if and only if $\Phi = \phi * \chi_{[0,1]}$. It is easy to see that $\chi_{[-1,0]} * \tilde{\Phi} = \tilde{\phi}$ if and only if $\tilde{\phi}'(x) = \tilde{\Phi}(x+1) - \tilde{\Phi}(x)$. Then it is easy to evaluate the function $\tilde{\Phi}(x)$, provided that $\tilde{\Phi}$ has compact support and the value of $\tilde{\phi}'(x)$ is known.

The following proposition shows the beautiful relation between the elevated wavelet Ψ and the original wavelet ψ. That is

$$\Psi(x) = 4\int_{-\infty}^{x} \psi(t)\, dt \quad \forall x \in \mathbb{R}. \tag{3.10}$$

Proposition 3.6. *Let $\Phi(x)$ be the elevation of the orthonormal scaling function $\phi(x)$ and $\tilde{\Phi}(x)$ is defined by (3.9). If we construct the wavelet $\Psi(x)$ from $m_1^{\Phi}(\omega)$, then $\{\Psi'(x-n)\}$ is an orthogonal set.*

Let $\phi(x)$ be the Daubechies orthonormal scaling function of order p. If we define $\phi_r(x)$ by

$$\phi_r(x) = \phi * \chi_{[0,1]} * \ldots * \chi_{[0,1]} \quad \text{(r times)},$$

for $1 \leq r < p$. Then, if $\{\phi_r, \psi_r, \tilde{\phi}_r, \tilde{\psi}_r\}$ is the biorthogonal system as chosen in this section, we have

$$\psi_r^{(r)}(x) = 4^r \psi(x).$$

That is, $\{\psi_r^{(r)}(x-n)\}$ is an orthogonal set.

Now we are at the position to prove that, under this setting, $\{\Phi(x-n\}$ are not orthogonal. First, we introduce the following lemma.

LEMMA 3.1. *Let $\phi(x)$ be an orthonormal scaling function and $\psi(x)$ be the associated orthonormal wavelet. If both ϕ and ψ are in $\mathcal{C}^p$ for some $p > 0$, then*

$$\langle \psi^{(r)}(x), \psi^{(s)}(x-n) \rangle = 2^{r+s} \langle \phi^{(r)}(x), \phi^{(s)}(x-2n) \rangle - \langle \phi^{(r)}(x), \phi^{(s)}(x-n) \rangle,$$

where $1 \leq r, s \leq p$.

PROPOSITION 3.7. *Let $\phi(x)$ be an orthonormal scaling function, and $\Phi(x)$ is the elevation of $\phi(x)$. Then $\{\Phi(x-n)\}$ is not orthogonal.*

Proof: If $\{\Phi(x-n)\}$ were orthogonal, we can modify the functions by a scalar to make them orthonormal. And note that Φ and Ψ are in $\mathcal{C}^1$. Then, apply the previous lemma with $r = s = 1$, we have

$$\begin{aligned} \delta_{0,n} &= \langle \Psi'(x), \Psi'(x-n) \rangle \\ &= 4\langle \Phi'(x), \Phi'(x-2n) \rangle - \langle \Phi'(x), \Phi'(x-n) \rangle \\ &= \langle \Phi'(x), \Phi'(x-n) \rangle. \end{aligned}$$

But it is easy to check that

$$\langle \Phi'(x), \Phi'(x-n) \rangle = \begin{cases} 2 & \text{if } n = 0, \\ -1 & \text{if } n = 1 \text{ or } -1, \\ 0 & \text{otherwise.} \end{cases}$$

This is a contradiction, so $\{\Phi(x-n)\}$ cannot be orthogonal.

At last, we present two simple algorithms. Let $\phi(x)$ be an orthonormal scaling function, $\psi(x)$ be the associated orthonormal wavelet, and $\{c_n\}_{n=0}^{N-1}$ be the nonzero scaling coefficients. Suppose $\phi(x)$ has the approximation degree p, then $N = 2p$. Let $\Phi(x)$ and $\tilde{\Phi}(x)$ be the pair of elevated scaling functions defined in the previous section. Let $\{h_n\}$, $\{\tilde{h}_n\}$ be the scaling coefficients of $\Phi(x)$ and $\tilde{\Phi}(x)$, respectively. By (3.3), we can easily derive the following algorithm.

Algorithm 1.

$$h_0 = \frac{c_0}{2}, \;\; h_N = \frac{c_{N-1}}{2}$$

$\texttt{FOR}\ i = 1\ \texttt{TO}\ N-1$

$$h_i = \frac{c_{i-1} + c_i}{2}$$

$\texttt{END}$

For $\tilde{h}_n$, we compare the coefficients of $m_0^{\tilde{\Phi}}$ and m_0^{ϕ}. For instance, see (3.8). Here we assume the practical case such that $m_0^{\phi}(\omega)$ has at least a factor of $(1+e^{-i\omega})/2$. The following algorithm derives $\tilde{h}_n$ from c_n.

Algorithm 2.

$$\tilde{h}_{-1} = 0$$

$\texttt{FOR}\ i = 0\ \texttt{TO}\ N-2$

$$\tilde{h}_i = 2c_i - \tilde{h}_{i-1}$$

$\texttt{END}$

By (3.7) we observe that the number of vanishing moments of Ψ is one more than that of ψ. Conversely, by (3.8), we observe that the number of vanishing moments of $\tilde{\Psi}$ is one less than that of ψ. By (3.5), we can also see that Φ has higher regularity than that of ϕ; and by (3.9), we see that $\tilde{\Phi}$ has less regularity than that of ϕ. The scaling coefficients $\tilde{h}_n$ of $\tilde{\Phi}$ has the similar property as that for h_n stated in Proposition 3.2.

Proposition 3.8. *Let $\phi(x)$ be defined as in* (2.1) *and c_n satisfy the condition* (A) *in* (3.1). *If $\tilde{\Phi}(x)$ is defined by $\chi_{[-1,0]} * \tilde{\Phi} = \phi$, then the scaling coefficients $\tilde{h}_n$ of $\tilde{\Phi}(x)$ satisfy*

$$\sum_{n=0}^{2p-2} (-1)^n n^m \tilde{h}_n = 0, \;\; \textit{where } m = 0, \ldots, p-2 \quad (0^0 := 1).$$

4. Generalizations. Let us now turn to the general setting and assume that $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ is a biorthogonal system. Even if $\phi(x)$ is not an orthonormal scaling function (and $\psi(x)$ is not an orthonormal wavelet), many properties of $\Phi(x)$ (the elevation of $\phi(x)$) and $\Psi(x)$ can still be preserved. For example, $\Phi(x)$ is still a scaling function and still generates a multiresolution analysis of $L^2(\mathbb{R})$; the approximation degree of $\Phi(x)$ is better than $\phi(x)$, etc. Specifically, all theorems and propositions hold without modifications, except for those mentioned below. ϕ would be changed to $\tilde{\phi}$ in (3.9), and (3.10) would be still true but $\Psi'(x)$ would not be orthogonal. The dual functions $\{\tilde{\Phi}, \tilde{\Psi}\}$ would be constructed in the same way as described in Section 2. All of the proofs remain the same with small modifications. Algorithm 1 would hold without modification, Algorithm 2 and PROPOSITION 3.8 would hold with c_n changed to $\tilde{c}_n$ and ϕ changed to $\tilde{\phi}$.

From equation (3.9), we see that the regularity of $\tilde{\Phi}$ is worse then that of $\tilde{\phi}$. One advantage of starting the elevation scheme from a biorthogonal system, instead of an orthonormal system, is that we can get a smoother biorthogonal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$. If we do not need the property of orthogonality for $\{\Psi'(x-n)\}$, we can apply the lifting scheme to (1) first elevate from $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ to $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$, then lift $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$; or (2) first lift $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$, then elevate the lifted biorthogonal system to $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$. In either way, we can raise regularity of $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$.

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