ℓ^2 IS NOT CLOSED UNDER CONVOLUTION*

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This article is written to faculty. We hope it will be helpful to colleagues who are preparing wavelets or Fourier analysis or DSP related applied math courses for undergraduate students.

1. Background.

In recent years, more and more colleagues around the world put their efforts on lowering the doorsill of mathematics for Fourier analysis and wavelets in particular, or digital signal processing (DSP) in general, and bring the related material into undergraduate curriculum. We coined various course titles to identify the field, to convey the idea and contents of the course, and to attract students' attention. For instance, one of the authors constructed a course *Computational Fourier Analysis* in his school, and Prof. Wickerhauser constructed a course *Mathematics for Multimedia* [1] in Washinton University, St. Louis. Compared to more *traditional* "applied math" such as differential equations and numerical analysis, the DSP related mathematics course is a new dish on the menu. We are less experienced in what material to collect and how to organize them, and textbooks written for undergraduates, such as *Wavelets and Filter Banks* by Strang and Nguyen [2], are just beginning to emerge.

This article derived from a particular situation happened in the course *Computational Fourier Analysis*. After the students were introduced to the discrete and continuous convolution, and they knew that the (continuous) convolution of two square integrable functions may not be square integrable, one asked whether or not the (discrete) convolution of two square summable sequences is square summable. This was a terrific question and it is a shame if we left it unproved. But in such a 3-credit course we only want to spend 15 hours to bring the students from one year of experience in calculus and linear algebra up to this level of abstraction. So the problem is how to answer this question (with proof) quickly with only the material that students have learned so far, in this course. We found a solution, and better yet, the proof involved the techniques and functions that were introduced in this course and would appear frequently in this course, therefore the question (and solution) fit naturally into the curriculum. We share this experience with our colleagues in this short article.

In the following, we will sketch the path by which we get to this point. Then we will list the

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facts that students have learned thus far. And finally we show that there is indeed a sequence $\boldsymbol{c} \in \ell^2$ such that $\boldsymbol{c} * \boldsymbol{c} \notin \ell^2$.

2. The syllabus.

In the course *Computational Fourier Analysis*, we start with a review of Taylor expansion and concepts of vector spaces like basis, inner product, norm and projection. Then we introduce the *box* function (the characteristic function of $[0,1)$), from which we will later derive B-splines and Haar wavelets. By discussing the approximation of continuous functions by step functions, we introduce the dilation and translation of the box function, the idea of multiresolution analysis, and the idea of distance, norm, and inner product between continuous functions in a closed interval. Then, naturally, the concept of a functional linear space will appear.

We then bring it to the students' attention that the continuity is not essential to the properties and concepts that we have developed so far. At this point, we hope the students have enough motivation to consider square integrable functions and the linear space formed by them. By examining the analog between the L^2 space and the $\mathbb{R}^2$ space, we point out that there is one thing missing, namely the completeness. But we tell students honestly that this is a topic we cannot afford to explore in this course, and give them directions if they are still interested to know. Then we go on to Fourier series, treating it as an orthonormal decomposition in the $L^2(T)$ space. This is the place where square summable sequences (ℓ^2) are introduced. By a Riemman-sum argument, Fourier series is informally extended to Fourier transform. We have to introduce the Plancherel theorem and Parseval's identity, because they are essential for understanding the time-frequence analysis. But here is another point where we have to sacrifice formal proofs; instead, we give heuristic examples (analogous to the Fourier serious results). This is the place where L^2 and ℓ^2 are compared. Later on, the finite energy signal functions and digital signals will reside in L^2 and ℓ^2 respectively.

After Fourier series and Fourier transform are introduced, we define the discrete and continuous convolution (motivated by the multiplication of trigonometric polynomials). Then we define B-splines by the convolution of box functions. The *hat* function comes up as an example, and we explore some properties of B-splines which are analogous to those of wavelets and which are useful later.

Since it is easy and worth the trouble to point out that the multiplication of two L^2 functions may not be in L^2, so we include this in the curriculum. The example we used in the classroom was $x^{-\frac{1}{3}} \cdot x^{-\frac{1}{3}}$ in the interval $(0,1)$. As we derive a basic fact that the Fourier transform of convolution equals the multiplication of Fourier transforms, and together with the Parseval's identity it is easy to see that the convolution of two L^2 functions may not be in L^2, therefore we include this in the curriculum as well. Here comes the point to say that the convolution of two ℓ^2 sequences may not be in ℓ^2.

3. List of known facts.

Now, let us list what are known to the students. They know the definition of following linear spaces: $L^2(\mathbb{R})$ consists of functions $f(x)$ defined on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} |f|^2\, dx$ exists, $L^2(\mathbb{T})$ consists of 2π-periodic functions $f(x)$ defined on $\mathbb{R}$ such that $\int_{-\pi}^{\pi} |f|^2\, dx$ exists, and ℓ^2 consists of sequences $\boldsymbol{c} = (c_k)$, $k \in \mathbb{Z}$, such that $\sum |c_k|^2$ exists. As noted above, the integrals are realized in the Riemann sense. They also know that $\{\frac{1}{\sqrt{2\pi}}\, e^{ikx} \mid k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{T})$, and for any continuous $f(x) \in L^2(\mathbb{T})$ we have a pointwise convergent Fourier series

$$f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k\, e^{ikx}.$$

Here $(c_k) = \boldsymbol{c} \in \ell^2$ and $\sum |c_k|^2 = 2\pi \int_{-\pi}^{\pi} |f|^2\, dx$. They are told the convergence theorem for piecewise smooth functions, and they are shown the Gibb's phenomenon. They have done an exercise to show that if $f(x)$ is an even function then c_k are real and $c_k = c_{-k}$.

Let $f, g \in L^2(\mathbb{R})$ and $\boldsymbol{u}, \boldsymbol{v} \in \ell^2$, the continuous and discrete convolution are defined respectively as

$$[f * g](x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy$$

$$(\text{the element of index } k \text{ for } \boldsymbol{u} * \boldsymbol{v}) = \sum_{n=-\infty}^{\infty} u_{k-n} v_n.$$

Let the Fourier transform of $f(x)$ be defined by

$$\mathcal{F}[f(x)](\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

then $\|\hat{f}\|^2 = 2\pi \|f\|^2$ and $\mathcal{F}[f * g] = \hat{f} \cdot \hat{g}$.

The students are familiar with the box function $\chi_{[0,1)}$. For our convenience we denote the centered box function $\chi_{[-\frac{1}{2},\frac{1}{2})}$ by $\phi(x)$. Let $\Phi(x) = \big[\phi * \phi\big](x)$, and $\Phi(x)$ is called hat function. They have done some exercises to know that

$$\hat{\phi}(\omega) = \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}, \qquad \mathcal{F}[\phi(x-k)](\omega) = e^{-ik\omega}\, \hat{\phi}(\omega)$$

and

$$\Phi(x-k) = \big[\phi(x+k) * \phi(x)\big](x) = \big[\phi(x) * \phi(x-k)\big](x).$$

Students learned that $\{\phi(x-k)\}$ is an orthonormal set, but $\{\Phi(x-k)\}$ is not. They are asked to calculate that

$$\int_{-\infty}^{\infty} \Phi(x)\Phi(x-k)\, dx = \begin{cases} \frac{2}{3} & \text{if } k = 0, \\ \frac{1}{6} & \text{if } k = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

With these ingredients, we are going to show that there is a real sequence $\boldsymbol{c} \in \ell^2$ such that $\boldsymbol{c} * \boldsymbol{c} \notin \ell^2$.

4. The theme.

We are motivated by the observation that if $g(x) = \sum c_k \phi(x-k)$, then formally

$$\hat{g}(\omega) = \sum c_k e^{-ik\omega}\, \hat{\phi}(\omega) = 2\pi f(-\omega)\, \frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}$$

for some $f(\omega) \in L^2(\mathbb{T})$. If we choose

$$f(\omega) = |\omega|^{-\frac{1}{3}}\, \frac{\frac{\omega}{2}}{\sin\frac{\omega}{2}}$$

for $\omega \in [-\pi, \pi]$, define arbitrarily for $\omega = 0$ and extend it periodically to $\omega \in \mathbb{R}$. Then $f(\omega)$ is an even function in $L^2(\mathbb{T})$ and $\boldsymbol{c} = (c_k)$ is a real even sequence in ℓ^2. Since $\|g\|^2 = \sum c_k^2 < \infty$, so g and $\hat{g}$ are in $L^2(\mathbb{R})$ and

$$\hat{g}(\omega) = 2\pi\, |\omega|^{-\frac{1}{3}}, \quad \text{for } \omega \in [-\pi, \pi].$$

Since $g(x)$ can also be written as $\sum c_k \phi(x+k)$, we have

$$\begin{aligned}
\big[g * g\big](x) &= \Big[\big(\sum c_k \phi(x+k)\big) * \big(\sum c_n \phi(x-n)\big)\Big](x) \\
&= \sum_k \sum_n c_k\, c_n \big[\phi(x+k) * \phi(x-n)\big](x) \\
&= \sum_k \sum_n c_k\, c_n \big[\phi(x) * \phi(x-k-n)\big](x) \\
&= \sum_k \sum_n c_k\, c_n \Phi(x-k-n) = \sum_k \sum_m c_k\, c_{m-k} \Phi(x-m) \\
&= \sum_m \Big(\sum_k c_{m-k} c_k\Big)\, \Phi(x-m) =: \sum_m d_m \Phi(x-m)
\end{aligned}$$

Here $(d_n) = \boldsymbol{d} = \boldsymbol{c} * \boldsymbol{c}$. By formally using the Schwarz inequality, we have

$$\int_{-\infty}^{\infty} \Big|\big[g * g\big](x)\Big|^2\, dx = \frac{1}{6}\Big(\sum_n d_n d_{n+1} + 4\sum_n d_n^2 + \sum_n d_{n-1} d_n\Big) \le \sum_n d_n^2.$$

But

$$\int_{-\infty}^{\infty} \Big|\big[g * g\big](x)\Big|^2\, dx = \frac{1}{2\pi}\|\hat{g} \cdot \hat{g}\|^2 \ge 2\pi \int_{-\pi}^{\pi} |\omega|^{-\frac{4}{3}}\, d\omega = \infty$$

so we see that $\boldsymbol{c} * \boldsymbol{c} \notin \ell^2$.

If the class should worry about the justice of using the Schwarz inequality before examining whether $\boldsymbol{d}$ is in ℓ^2 or not, then we have to go through the partial-sum and taking-the-limit argument. We did not go into such a detail in our class.

REFERENCES

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