WAVELETS AND TWO-POINT BOUNDARY VALUE PROBLEMS*

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Abstract. We develop a stable direct method with linear complexity for solving large banded sparse linear systems which are typical for the numerical solution of one-dimensional differential equations. This method was derived on the basic idea of multiresolutional decompositions that are carried out by the theory of wavelets.

1. Introduction. Because of their similarities with Fourier transforms and finite elements, wavelets are studied for the applicability of being a tool in scientific computation and numerical analysis, such as the numerical solution of differential equations. Among the recent articles on the later subject, we refer to the works by Glowinski, Lawton, Ravachol and Tenenbaum [1].

However, some authors found that the "usual" wavelets, which are orthogonal among the original functions instead of the first derivatives, do not efficiently localize the differential operator. Recently there is a new set of biorthogonal wavelets that diagonalize the differential operator. The L^2-orthonormality is helpful to represent "nice" integral operators like the Calderon-Zygmund operator, as demonstrated by Beylkin, Coifman and Rokhlin [2]. But the band-width will be wider when applied to differential operators, compared with the matrices induced by an approximation of same order in finite difference or finite element.

Besides the locality problem mentioned above, wavelets on the real line cause tedious problems for those "crossing" the boundary of a domain. Wavelets that "live" in intervals can be derived from those "live" on real lines by, basically, applying the Gram-Schmidt process on those crossing the boundary. They are first constructed by Meyer, then improved by Andersson, Hall, Jawerth and Peters [3]. The numerical experiences of these new wavelets are yet to be explored.

There are other works looking for alternatives that are orthogonal among the derivatives and are better adapted for bounded domains. Among them we refer to Xu and Shann [4]. In this article, we also do not directly use wavelets. We found that the fundamental ideas of locality, orthogonal among the derivatives and multiresolution decomposition are the essential parts. We explored the connections of these ideas and derived this class of direct methods for solving large banded and sparse linear systems, which are typical when a differential equation is discretized by finite element or finite difference methods.

2. State of problem. The second order finite element or finite difference approximation of the differential equation

$$-u'' = f \tag{1}$$

is basically of the form

$$-u_{j-1} + 2u_j - u_{j+1} = f_j.$$

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However, the bandedness may be affected by the associated boundary conditions. For instance the periodic condition with the normalization $\int u\,dx = 0$. Then the matrix induced by (1) is

$$\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}_{N\times N} . \tag{2}$$

If the system with matrix (2) is to be solved by Gaussian elimination, the last two equations will be

$$\begin{cases} \dfrac{N}{N-1}u_{N-1} - \dfrac{N}{N-1}u_N = f_{N-1} \\ \qquad\qquad\qquad Nu_N = f_N. \end{cases} \tag{3}$$

We shall solve u_N by (3), then solve backward for $u_{N-1}, \ldots, u_1$. In our case $|u_k|$ is independent of N and hence (3) is a typical sign of instability. For instance, if $N = 10^6$, then we will loss 6 significant digits in this calculation. This situation is generally resolved by the Gaussian elimination with pivoting. But clearly this is not a desired approach because we will loss the sparsity and hence increase the computational complexity up to $O(N^2)$.

The goal in our article is to solve this kind of problem by a direct method with $O(N)$ computational complexity, and yet covers a wide range of applications like the foregoing examples.

3. Basic idea. The basic idea is the multiresolutional decomposition. Suppose the numerical solution of a differential equation is done is a finite dimensional space V_L, and the solution is $u_L \in V_L$. Then we decompose

$$\begin{aligned} u_L &= u_{L-1} + (u_L - u_{L-1}) = u_{L-2} + (u_{L-1} - u_{L-2}) + (u_L - u_{L-1}) \\ &= \cdots := u_0 + w_0 + \cdots + w_{L-1}. \end{aligned}$$

Here u_ℓ is considered the resolution of u at level ℓ; while $w_\ell = u_{\ell+1} - u_\ell$ is the difference, or "detail," between the two resolutions. We want to design a sequence of spaces

$$V_0 \subset V_1 \subset \cdots \subset V_L$$

such that

$$V_\ell = V_0 + W_0 + \cdots + W_{\ell-1}.$$

Suppose u_ℓ is solved by the linear system $A_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell$; here $A_L = A$. After one step of decomposition, we shall have the linear system

$$\begin{pmatrix} A_{\ell-1} & B_{\ell-1} \\ C_{\ell-1} & D_{\ell-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\ell-1} \\ \boldsymbol{w}_{\ell-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{\ell-1} \\ \boldsymbol{g}_{\ell-1} \end{pmatrix}. \tag{4}$$

For simplicity we assume that A is symmetric, so $B_{\ell-1} = C_{\ell-1}^T$. For our purpose, we would like the decomposition to be done in spaces $V_{\ell-1}$ and $W_{\ell-1}$ such that, ideally, we hope (i) $B_{\ell-1} = 0$, (ii) $D_{\ell-1} = I$, and (iii) $A_{\ell-1}$ has the same banded diagonal terms as A_ℓ has.

Let $\phi_k^\ell(x)$ and $\psi_k^\ell(x)$ be the basis functions of V^ℓ and W^ℓ respectively. For the foregoing purpose we conclude that the following characteristics are important for ϕ_k^ℓ and ψ_k^ℓ: (1$'$) suppψ_k^ℓ intersects with only a few of suppϕ_k^ℓ, no more than the band-width of A; and we like to have $(\phi_k^\ell)'$ orthogonal to $(\psi_k^\ell)'$. (2$'$) suppψ_k^ℓ are mutually disjoint; or $(\psi_k^\ell)'$ are orthonormal. (3$'$) ϕ_k^ℓ are compactly supported to produce a sparse matrix.

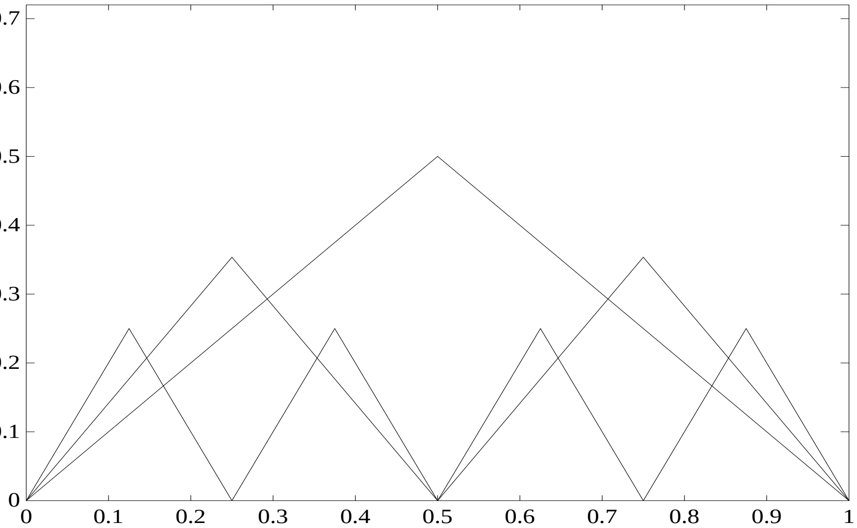

Figure 1. Piecewise linear hierarchical basis at levels $\ell = 0,\ 1,\ 2$.

4. Derivation from hierarchical bases. Now we suppose the differential equation is on the spatial domain $(0,1)$, on which we define a sequence of partitions. For simplicity of notations (which is not theoretically essential), we let $h^\ell = 2^{-\ell}$, for $\ell = 1, \ldots, L$; here $L > 0$ is the prescribed highest resolution. The degree of freedom for each level ℓ is $N_\ell = 2^\ell - 1$. Define the partitions $\mathcal{T}_\ell = \{x_k^\ell\}$ to be $x_k^\ell = k\,h^\ell$, where $k = 1, \ldots, N_\ell$. Let $\phi_k^\ell(x)$ be the piecewise linear nodal basis function associated with x_k^ℓ in $\mathcal{T}_\ell$. Suppose we want to obtain the Galerkin solution u_h in the highest resolution. That is, let $\mathcal{T}_h = \mathcal{T}_L$ and $V_h = \mathrm{span}\{\phi_k^L(x) : k = 1, \ldots, N_L\}$. We denote the nodal basis for V_h by $\mathcal{N}$.

The hierarchical basis associated with this set of nodal basis is defined by

$$\psi_k^\ell(x) = \sqrt{\frac{h^\ell}{2}}\phi_{2k-1}^{\ell+1}(x), \quad \text{for } \ell = 0, \ldots, \ell - 1 \text{ and } k = 1, \ldots, 2^\ell.$$

We denote the hierarchical basis by $\mathcal{H}$. One can verify that $|\mathcal{N}| = |\mathcal{H}|$ and $V_h = \mathrm{span}\mathcal{H}$. That is, $\mathcal{H}$ consists of another basis for V_h. The first levels of basis functions in $\mathcal{H}$ are plotted in Figure 1.

Observe that the first generalized derivatives of $\psi_k^\ell(x)$ are exactly the Haar functions on $[0,1]$. They form an orthonormal set of functions. Therefore, in the special case of (1) with Dirichlet boundary condition, the matrix A is the identity.

Here we do not view the hierarchical basis as another set of basis for V_h, neither change the basis from $\mathcal{N}$ to $\mathcal{H}$ all at once. Let V_ℓ be the subspace of V_h spanned by the nodal basis functions on $\mathcal{T}_\ell$, W_ℓ be that spanned by the hierarchical basis ψ_k^ℓ at the level ℓ. We focus on the relations between V_ℓ and $V_{\ell-1}$ and $W_{\ell-1}$. First of all, it is clear that $\dim V_\ell = \dim V_{\ell-1} + \dim W_{\ell-1}$.

We drop the scaling coefficient for ψ_k^ℓ at this moment. The correct scale can be restored later. Now we have

$$\phi_k^{\ell-1} = \tfrac{1}{2}\phi_{2k-1}^\ell + \phi_{2k}^\ell + \tfrac{1}{2}\phi_{2k+1}^\ell, \qquad \psi_k^{\ell-1} = \phi_{2k-1}^\ell.$$

Also, notice that

$$< (\psi_k^\ell)',\, (\psi_m^\ell)' >= \delta_{km}, \qquad < (\psi_k^\ell)',\, (\phi_m^{\ell-1})' >= 0.$$

Here $< \cdot, \cdot >$ is the L^2 inner product.

Since $V_\ell = V_{\ell-1} + W_{\ell-1}$, there is a change-of-basis matrix between the two sets of basis functions. That is, for any $u_\ell \in V_\ell$, we have

$$u_\ell = \sum u_k^\ell \phi_k^\ell(x) = \sum u_k^{\ell-1} \phi_k^{\ell-1}(x) + \sum w_k^{\ell-1} \psi_k^{\ell-1}(x).$$

There is a matrix to convert between $\{u_k^\ell\}$ and $\{u_k^{\ell-1}, w_k^{\ell-1}\}$. We illustrate the pattern of the matrix by the case of $\ell = 3$:

$$\begin{pmatrix} 1/2 & & & 1 & & & \\ 1 & & & & & & \\ 1/2 & 1/2 & & 1 & & & \\ & 1 & & & & & \\ & 1/2 & 1/2 & & 1 & & \\ & & 1 & & & & \\ & & 1/2 & & & & 1 \end{pmatrix} \begin{pmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \\ w_1^2 \\ w_2^2 \\ w_3^2 \\ w_4^2 \end{pmatrix} = \begin{pmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \\ u_4^3 \\ u_5^3 \\ u_6^3 \\ u_7^3 \end{pmatrix}.$$

Observe that $u_k^{\ell-1} = u_{2k}^\ell$. That is, when the solution u_h is obtained in $V_{\ell-1}$, the differences between that of V_ℓ and $V_{\ell-1}$ are totally presented in $W_{\ell-1}$. This feature makes the reconstruction easier.

With this change of basis, we can solve the lienar system $A\boldsymbol{u} = \boldsymbol{f}$ where A is given as in (2) by the decomposition as given in (4). All three wishes under (4) can be made true. And thus we have a direct method of computational complexity $O(N)$, which is also stable.

5. Conclusion. The "usual" wavelets do not show much advantage for solving differential equations. What essential are the concepts of orthogonal on the derivative level, the multiresolutional decomposition and the compact supportness (locality) which can also fit the boundary conditions.

Here we demonstrate a result in this way of thinking. The idea of multiresolution is used to reduce the linear system to half size. If the band-width of the matrix is independent of N, usually this means the problem is one-dimensional, this method gives a direct method of complexity $O(N)$ and it is stable for many applicable physical problems.

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