# THE LEVERAGED WAVELETS AND GALERKIN-WAVELETS METHODS*

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**Abstract.** We present a scheme that leverage orthonormal or biorthogonal wavelets to a new system of biorthogonal wavelets. The leveraged biorthogonal wavelets will have some nice properties. If we start with orthonormal wavelets, the leveraged scaling functions and wavelets are compactly supported and are differentiable. The derivatives of the leveraged wavelets are orthogonal to their translations; the derivatives of the leveraged scaling functions are nearly orthogonal to their translations; and the derivatives of the leveraged scaling functions and wavelets are orthogonal to each other. This feature may be valuable for the numerical solution of differential equations. If we start with B-splines and cooperating with the lifting scheme of Sweldens, our leverage scheme can reproduce all of those biorthogonal wavelets by Cohen, Daubechies and Feauveau. There is a simple algorithm to calculate new filter coefficients from the old filter coefficients. In the end of this article we test the newly constructed biorthogonal wavelets in the framework of Galerkin-wavelet methods.

**Key words.** biorthogonal wavelets, regularity, convolution, approximation degree, stiffness matrix, condition number.

**1. Motivation.** The motivation of this work is the following formula

$$\Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt \tag{1.1}$$

that was suggested by Xu and Shann in their paper on Galerkin-wavelets methods [17]. Here $\phi(x)$ is an orthonormal scaling function. In that paper, they were studying the Galerkin basis functions derived from orthonormal wavelets that will produce well-conditioned stiffness matrices and will satisfy the Dirichlet boundary conditions. The formula (1.1) was suggested to ensure the compact supportedness of the Galerkin-wavelets basis functions. We follow this hint and discovered that it is actually the starting point of a general scheme to derive from a wavelet to another wavelet with some better properties. We call it the *leverage scheme* of biorthogonal wavelets.

Following the spirit of the Galerkin-wavelets methods [17], the ultimate goal for the leverage scheme is the application for numerical solution of differential equations.

In Section 2, we will briefly review the fundamentals and define the notations. In Section 3, we will define the leverage scheme and present the main properties for the orthonormal case. As a preview, let $\phi(x)$ and $\psi(x)$ be a pair of orthonormal scaling function and wavelet. Let $\Phi(x)$ be the leverage of $\phi(x)$ as defined in (1.1). Then $\Phi(x)$ is again a scaling function and it generates a multiresolution analysis (MRA) of $L^2(\mathbb{R})$. If $\phi(x)$ has the approximation degree $p$, then $\Phi(x)$ will have the approximation degree $p+1$. The derivatives $\Phi'(x-k)$ are almost orthogonal in the sense that $\langle \Phi'(x), \Phi'(x-k)\rangle = 0$ for all $|k| \geq 2$; the nonzero terms are always $\{-1, 2, -1\}$ (for all $p$) which reminds us the tridiagonal stiffness matrix formed by piecewise linear finite element basis. But $\Phi(x)$ cannot be an orthonormal scaling function. We must construct a system of biorthogonal wavelets from $\Phi(x)$, namely $\{\Phi(x), \Psi(x), \tilde{\Phi}(x), \tilde{\Psi}(x)\}$. We will see that

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$\Psi'(x) = 4\psi(x)$, therefore (1) $\Psi'(x-k)$ are orthogonal to each other and (2) $\Phi'(x)$ and $\Psi'(x)$ are orthogonal to each other.

It is also very interesting to realize that, as a by-product, the leverage scheme and the *lifting scheme* of Sweldens [15] can work together to re-produce all of the compactly supported biorthogonal wavelets by Cohen, Daubechies and Feauveau [5]. This procedure shall also give us a better understanding of these useful wavelets. We will demonstrate this phenomenon in Section 4.

The rest of this section is devoted to a brief background explanation on why we choose this approach to the application of wavelets for numerical solution of differential equations.

Since the introduction of *wavelets* ten years ago, there are intensive research activities in this area, in both theoretical and applied aspects. For more introductory details, we would like to refer the readers to [7, 12, 14]. Among the applications, there are the numerical solution for integral equations and differential equations. Wavelets based numerical methods for the numerical solution of integral equations are reasonably successful—at least, for those with kernels which do not oscillate too much. For this, people basically follow the work of [1]. On the other hand, wavelets based numerical methods for the numerical solution of differential equations are not as successful as those for integral equations. There are different points of views on why it might be difficult and how it might be resolved. Some groups of researchers share the same point of view that there must be the "right" wavelets which are "made" for the differential equations. For instance, Jaffard [8], Xu and Shann [17], Jawerth and Sweldens [9], Liandrat and Tchamitchian [10], Chern and Shann [2], Ponenti and Liandrat [11] are among these groups. All of the above works except for [2] (which is more like a finite difference method) are based on the Galerkin method to discretize the differential equations.

We may not have the full answer (yet) about what make wavelets attractive to the numerical solution of differential equations. But there are some plausible reasons. The wavelets have simple structures, they have good approximation ability, they have compact or "short" supports, they can localize functions in both time and frequency domain, and there is a fast algorithm to change bases between high and low resolutions.

The biorthogonal system $\{\Phi(x), \Psi(x), \tilde{\Phi}(x), \tilde{\Psi}(x)\}$ has some promising properties that make it probably the "right" basis for the Galerkin-wavelets methods. First of all, they have all the "standard" merits of wavelets that are listed above. In addition, the short supports of $\Phi(x)$ and $\Psi(x)$ ensure the *sparse* discretized matrices, while the biorthogonal structure and the properties of $\Phi'(x)$ and $\Psi'(x)$ make it possible to efficiently change the bases for *well-conditioned* matrices. Furthermore, we think the biorthogonal structure makes it possible for designing multigrid-like algorithms.

A preliminary version of this manuscript was published in the International Wavelets Conference in Tangier, Morocco [13]. The numerical tests for Galerkin-wavelets methods are not completely satisfied yet, and so this is still an incomplete manuscript.

**2. Fundamentals.** The purpose of this section is to define the notation and terminology for the rest of this article. For a comprehensive introduction of wavelet theory, we would like to refer the readers to [3, 7, 14].

Let $\phi(x)$ be a *scaling function* which satisfies a finite (two-scale) scaling equation

$$\phi(x) = \sum c_k \phi(2x - k). \tag{2.1}$$

Here $c_k$ are called the *scaling coefficients* and we assume that only $c_0, c_1, \cdots, c_N$ are non-zero. Since we are only interested in the case that $\phi(x)$ being a low-pass filter, so we assume that $\hat{\phi}(0) = 1$. Here

$$\hat{\phi}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-ix\omega}\, dx$$

is the Fourier transform of $\phi(x)$. Let $c_0, \ldots, c_N \in \mathbb{R}$ with $\sum c_n = 2$, then there is a unique compactly supported distribution $\phi(x)$ which satisfies (2.1) with $\hat{\phi}(0) = 1$ and $\text{supp}\phi(x) = [0, N]$.

Let $p \geq 1$ be an integer. We say that $\phi(x)$ has an *approximation degree* $p$ if linear combinations of $\phi(x-k)$ can (locally) express polynomials of degree 0 through $p-1$. In fact, if the scaling coefficients $c_k$ satisfy $\sum(-1)^k k^m c_k = 0$, for $0 \leq m \leq p-1$, and the scaling function satisfies $\sum \phi(x-k) = 1$, for all $x \in \mathbb{R}$, then $\sum k^m \phi(x-k)$ is a polynomial of degree $m$ in any bounded interval $[a, b]$.

Define

$$m_0^{\phi}(\omega) = \frac{1}{2} \sum c_k e^{-ik\omega}$$

and we call it the *frequency response* function for $\phi(x)$. We will use the simplified notation $m_0(\omega)$ when there is no ambiguity in the context. Note that $m_0(\omega)$ is a trigonometric polynomial and it is $2\pi$-periodic. We say that $\phi(x)$ is an *orthonormal scaling function* if $\{\phi(x-k)\}$ forms an orthonormal set (in $L^2(\mathbb{R})$). In this case,

$$\sum |\hat{\phi}(\omega + 2k\pi)|^2 = 1 \quad \text{and} \quad |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1, \quad \forall \omega \in \mathbb{R}.$$

Let $V_j$ be the subspaces of $L^2(\mathbb{R})$ spanned by $\{\sqrt{2^j}\phi(2^j x - k)\}_k$. If there are constants $0 < A \leq B < \infty$ such that

$$A \leq \sum |\hat{\phi}(\omega + 2k\pi)|^2 \leq B,$$

then $\{\phi(x-k)\}$ forms a Riesz basis of $V_0$. Note that if $\phi(x)$ is an orthonormal scaling function, then $\{\phi(x-k)\}$ forms an orthonormal basis (which is a special case of Riesz basis with $A = B = 1$) of $V_0$. If $\{\phi(x-k)\}$ is a Riesz basis, then

$$\bigcap V_j = \{0\} \quad \text{and} \quad \overline{\bigcup V_j} = L^2(\mathbb{R}).$$

We say that $\phi(x)$ generates a *Multiresolution Analysis* (MRA) $\{V_j\}$ of $L^2(\mathbb{R})$.

Let $\phi \in L^2(\mathbb{R})$ be an orthonormal scaling function with approximation degree $p$ and it generates an MRA $\{V_j\}$. By (2.1), $V_j \subset V_{j+1}$. Let $W_j$ be the orthogonal complement of $V_j$ in $V_{j+1}$. Then $\oplus W_j = L^2$. It turns out that

$$\psi(x) = \sum (-1)^k c_{1-k} \phi(2x - k)$$

is in $W_0$ and actually $\{\sqrt{2^j}\psi(2^j x - k)\}_k$ is an orthonormal basis of $W_j$. This $\psi(x)$ is called an *orthonormal wavelet* associated with the scaling function $\phi(x)$. $\psi(x)$ is a high-pass filter and its frequency response function is

$$m_1^{\phi}(\omega) = -e^{-i\omega}\overline{m_0^{\phi}(\omega + \pi)}.$$

Note that $|m_0(\omega)|^2 + |m_1(\omega)|^2 = 1$ and $\hat{\psi}(\omega) = m_1(\frac{\omega}{2})\,\hat{\phi}(\frac{\omega}{2})$. Since $W_0 \perp V_0$, the first $p$ moments of $\psi(x)$ will vanish. That is, $\int x^m \psi\, dx = 0$, for $0 \leq m \leq p-1$.

A family of such compactly supported orthonormal scaling functions and wavelets was discovered by Daubechies [6] for each $p \geq 1$. In this case, $N = 2p - 1$ and the regularity is $\phi \in C^{\sigma(p)}$ where $\sigma(p)$ is asymptotically $\frac{1}{3}p$ when $p$ is large.

If $\phi(x)$ is not an orthonormal scaling function, there are mainly two ways to construct the (more generalized) associated wavelets. One is the semi-orthogonal approach by Chui and Wang [4], another is the biorthogonal approach by Cohen, Daubechies and Feauveau [5]. Both of them use the B-splines as the (non-orthonormal) scaling functions. We will follow the biorthogonal approach.

Let $\phi(x)$ be a scaling function which generates an MRA $\{V_j\}$, and $\tilde{\phi}(x)$ be another scaling function which generates another MRA $\{\tilde{V}_j\}$. More precisely, $\{\phi(x-k)\}$ is a Riesz bases for $V_0$ and $\{\tilde{\phi}(x-k)\}$ is a Riesz bases for $\tilde{V}_0$. Let $c_0, \cdots, c_N$ be the scaling coefficients for $\phi(x)$ and the frequency response functions $m_0^{\phi}(\omega)$ and $m_1^{\phi}(\omega)$ be defined as above. Let $\tilde{c}_0, \cdots, \tilde{c}_{\tilde{N}}$ be the scaling coefficients for $\tilde{\phi}(x)$, we can define the frequency response functions $m_0^{\tilde{\phi}}(\omega)$ and $m_1^{\tilde{\phi}}(\omega)$ analogously. If we can construct these functions such that

$$\widetilde{M}(\omega)\,\overline{M^t(\omega)} = 1, \tag{2.2}$$

then we will have a *biorthogonal system* $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ in the sense that

$$\begin{aligned} &\langle \tilde{\phi}(x-k), \phi(x-\ell)\rangle = \delta_{k\ell} \quad &&\langle \tilde{\psi}(x-k), \psi(x-\ell)\rangle = \delta_{k\ell} \\ &\langle \tilde{\psi}(x-k), \phi(x-\ell)\rangle = 0 &&\langle \tilde{\phi}(x-k), \psi(x-\ell)\rangle = 0 \end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ is the $L^2$ inner product, $\delta_{k\ell}$ is the Dirac delta notation,

$$M(\omega) = \begin{pmatrix} m_0^{\phi}(\omega) & m_0^{\phi}(\omega+\pi) \\ m_1^{\phi}(\omega) & m_1^{\phi}(\omega+\pi) \end{pmatrix} \quad \text{and} \quad \widetilde{M}(\omega) = \begin{pmatrix} m_0^{\tilde{\phi}}(\omega) & m_0^{\tilde{\phi}}(\omega+\pi) \\ m_1^{\tilde{\phi}}(\omega) & m_1^{\tilde{\phi}}(\omega+\pi) \end{pmatrix}.$$

Actually, $m_1^{\phi}(\omega)$ and $m_1^{\tilde{\phi}}(\omega)$ satisfy (2.2) if and only if they are of the form

$$m_1^{\tilde{\phi}}(\omega) = -e^{-i\omega}\,\overline{m_0^{\phi}(\omega+\pi)}\,s(2\omega) \quad \text{and} \quad m_1^{\phi}(\omega) = -e^{-i\omega}\,\overline{m_0^{\tilde{\phi}}(\omega+\pi)\,s(2\omega)},$$

where $s(\omega)$ is some trigonometric polynomial bounded away from 0. The simplest choice is that $s(\omega)=1$. In this case,

$$\psi(x) = \sum(-1)^k \tilde{c}_{1-k}\phi(2x-k) \quad \text{and} \quad \tilde{\psi}(x) = \sum(-1)^k c_{1-k}\tilde{\phi}(2x-k). \tag{2.3}$$

**3. Main Results.** Throughout the section, we let $\phi(x)$ be a compactly supported orthonormal scaling function of approximation degree $p$, and let $\psi(x)$ be the orthonormal wavelet associated with $\phi(x)$. Repeat (1.1) here, let

$$\Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\,dt \tag{3.1}$$

and we call $\Phi(x)$ the *elevation* of $\phi(x)$. We will show that $\Phi(x)$ is still a scaling function that generates a multiresolution analysis of $L^2(\mathbb{R})$. Therefore, there is a wavelet $\Psi(x)$ associated with $\Phi(x)$. Neither $\Phi(x)$ nor $\Psi(x)$ will be orthonormal. But $\Psi'(x) = 4\psi(x)$, so $\{\Psi'(x-k)\}$ is an orthogonal set and they are orthogonal to $\Phi'(x)$. Since $\Phi(x)$ is not an orthonormal scaling function, we will construct a pair of functions $\tilde{\Phi}(x)$, and $\tilde{\Psi}(x)$ such that $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$ forms a system of biorthogonal wavelets. With the biorthogonal structure, we will have the fast algorithm to change the bases between $V_{j+1}$ and $V_j + W_j$.

By (3.1), it is easy to check that $\Phi(x)$ satisfies a scaling equation

$$\Phi(x) = \sum_{k=0}^{2p} h_k \Phi(2x-k)$$

with $2p+1$ nonzero scaling coefficients $h_0, h_1, \ldots, h_{2p}$. Here

$$h_k = \frac{(c_{k-1}+c_k)}{2}, \tag{3.2}$$

note that $c_{-1} = c_{2p} = 0$. Furthermore, note that $\mathrm{supp}\Phi(x) = [0, 2p]$ and $\sum h_k = \sum c_k = 2$.

Since

$$\Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt = \int_{x-1}^{x} \phi(t)\, dt,$$

we have

$$\Phi(x) = \chi_{[0,1]} * \phi(x). \tag{3.3}$$

Here $*$ stands for the (continuous) convolution operator. Hence

$$\hat{\Phi}(\omega) = \widehat{\chi_{[0,1]}}(\omega)\hat{\phi}(\omega) = e^{-i\frac{\omega}{2}} \frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\, \hat{\phi}(\omega).$$

Note that $\hat{\Phi}(0) = \int \Phi\, dx = 1$ and $\hat{\Phi}(2k\pi) = 0$ for $0 \neq k \in \mathbb{Z}$.

The frequency response function for $\Phi(x)$ shall be defined by $m_0^{\Phi}(\omega) = \frac{1}{2}\sum h_k e^{-ik\omega}$. And we also have

$$m_0^{\Phi}(\omega) = (\frac{1}{2} + \frac{1}{2}e^{-i\omega}) m_0^{\phi}(\omega). \tag{3.4}$$

Therefore,

$$(h_n) = (\frac{1}{2}, \frac{1}{2}) * (c_n).$$

Here $*$ stands for the (discrete) convolution operator.

Proposition 3.1. *Let $h_k$ be determined by* (3.2) *and $c_k$ satisfy $\sum(-1)^k k^m c_k = 0$, for $0 \le m \le p-1$ (with $0^0 := 1$), then*

$$\sum_{k=0}^{2p} (-1)^k k^m h_k = 0, \textit{ for } 0 \le m \le p. \tag{3.5}$$

Proof: For $m = 0$, it follows from (3.2) that $\sum(-1)^k h_k = 0$. Assume that (3.5) holds for $m = 0, \ldots, \ell - 1$, where $0 < \ell \le p$. Then

$$\begin{aligned}
\sum_{k=0}^{2p} (-1)^k k^\ell h_k &= \sum_{k=0}^{2p} (-1)^k k^\ell (\frac{c_k + c_{k-1}}{2}) \\
&= \frac{1}{2} \sum_{k=0}^{2p-1} (-1)^k [k^\ell - (k+1)^\ell] c_k \\
&= \frac{1}{2} \sum_{k=0}^{2p-1} (-1)^{k+1} \sum_{n=0}^{\ell-1} \binom{\ell}{n} k^n c_k \\
&= -\frac{1}{2} \sum_{n=0}^{\ell-1} \binom{\ell}{n} \sum_{k=0}^{2p-1} (-1)^k k^n c_k \\
&= 0.
\end{aligned}$$

Hence, by induction, we have the proof. ∎

PROPOSITION 3.2. *If* $\sum \phi(x-k) = 1$, *then*

$$\sum_k \Phi(x-k) = 1, \quad \forall x \in \mathbb{R}. \tag{3.6}$$

Proof: For all $x \in \mathbb{R}$,

$$\begin{aligned}\sum_k \Phi(x-k) &= \sum_k \phi * \chi_{[0,1]}(x-k) \\ &= \int_{-\infty}^{\infty} \sum_k \phi(x-k-t)\chi_{[0,1]}(t)\,dt \\ &= \int_{-\infty}^{\infty} \chi_{[0,1]}(t)\,dt \\ &= 1.\end{aligned}$$

□

By (3.5) and (3.6), it follows that $\Phi(x-k)$ can span the polynomials $1, x, \ldots, x^p$ in any bounded interval. Specifically, $\sum k^m \Phi(x-k)$ is a polynomial of degree $m$, for $0 \le m \le p$.

Next, let $V_j = \text{span}\{\sqrt{2^j}\Phi(2^j x - k) \mid k \in \mathbb{Z}\}$. We would like to check if $\Phi(x-k)$ forms a Riesz basis of $V_0$. It amounts to show that there are constants $0 < A \le B$ such that

$$A \le \sum_{k=-\infty}^{\infty} |\hat{\Phi}(\omega + 2k\pi)|^2 \le B, \quad \forall \omega \in \mathbb{R}. \tag{3.7}$$

(See, for example, [4].) In order to do that, consider the $2\pi$-periodic function

$$\Gamma(\omega) := \sum_{k=-\infty}^{\infty} |\hat{\Phi}(\omega + 2k\pi)|^2, \quad \text{for } \omega \in \mathbb{R} \tag{3.8}$$

LEMMA 3.1. *If* $\phi \in L^1(\mathbb{R})$, *then* $\Gamma(\omega)$ *is a continuous function.*

Proof: Since $\phi \in L^1(\mathbb{R})$, so $\hat{\phi}(\omega)$ is continuous. Then

$$\hat{\Phi}(\omega + 2k\pi) = \left(\frac{\sin\frac{\omega+2k\pi}{2}}{\frac{\omega+2k\pi}{2}}\right) e^{-i\frac{\omega+2k\pi}{2}}\,\hat{\phi}(\omega + 2k\pi)$$

is also continuous. Define a sequence of functions $(\Gamma_n)$ by

$$\Gamma_n(\omega) = \sum_{k=-n}^{n} |\hat{\Phi}(\omega + 2k\pi)|^2 = \sum_{k=-n}^{n} \left|\frac{\sin\frac{\omega}{2}}{\frac{\omega+2k\pi}{2}}\right|^2 |\hat{\phi}(\omega + 2k\pi)|^2.$$

$\Gamma_k$ are continuous $2\pi$-periodic functions, and we shall check that $\Gamma_n \to \Gamma$ uniformly on $(-2\pi, 2\pi)$. Indeed, for $n \ge 2$,

$$\begin{aligned}|\Gamma_n - \Gamma| &= \left|\sum_{k=-\infty}^{-n-1} \left|\frac{\sin\frac{\omega}{2}}{\frac{\omega+2k\pi}{2}}\right|^2 |\hat{\phi}(\omega+2k\pi)|^2 + \sum_{k=n+1}^{\infty} \left|\frac{\sin\frac{\omega}{2}}{\frac{\omega+2k\pi}{2}}\right|^2 |\hat{\phi}(\omega+2k\pi)|^2\right| \\ &\le \sum_{k=-\infty}^{-n-1} \frac{4}{|\omega+2k\pi|^2}\|\phi\|_1^2 + \sum_{k=n+1}^{\infty} \frac{4}{|\omega+2k\pi|^2}\|\phi\|_1^2 \\ &\le \frac{2}{\pi^2}\|\phi\|_1^2 \sum_{k=n-1}^{\infty} \frac{1}{n^2} \le \frac{1}{3}\|\phi\|_1^2.\end{aligned}$$

Thus $\Gamma(\omega)$ is continuous on $(-2\pi, 2\pi)$, and hence on $\mathbb{R}$. □

LEMMA 3.2. *If there are $0 < A_\phi \le B_\phi$ such that $A_\phi \le \sum_k |\hat{\phi}(\omega + 2k\pi)|^2 \le B_\phi$, then $\Gamma(\omega) > 0$.*

Proof: If there is an $\omega_0 \in [0, 2\pi)$ such that $\Gamma(\omega_0) = 0$, then

$$\sum_{n=-\infty}^{\infty} \left| \frac{\sin \frac{\omega_0}{2}}{\frac{\omega_0 + 2k\pi}{2}} \right|^2 |\hat{\phi}(\omega_0 + 2\pi n)|^2 = 0.$$

If $\sin \frac{\omega_0}{2} = 0$, then $\omega_0 = 0$. But we know $\hat{\Phi}(0) = 1$ and $\hat{\Phi}(2k\pi) = 0$ for all $k \neq 0$, so $\Gamma(0) = 1$. Hence $\sin \frac{\omega_0}{2} \neq 0$ and it must be that $\hat{\phi}(\omega_0 + 2k\pi) = 0$, for all $k \in \mathbb{Z}$. This contradicts the assumption $0 < A_\phi \le \sum |\hat{\phi}(\omega_0 + 2k\pi)|^2$. So $\Gamma(\omega) > 0$ for $\omega \in [0, 2\pi)$, and hence for $\omega \in \mathbb{R}$. □

By the above lemmas, it is easy to prove (3.7) and consequently we have the following conclusion.

THEOREM 3.3. *If $\phi \in L^1 \cap L^2$ is a scaling function and $\{\phi(x - k)\}$ is a Riesz basis, then $\{\Phi(x - k)\}$ is also a Riesz basis and $\Phi(x)$ generates a multiresolution analysis of $L^2(\mathbb{R})$.*

When $\phi(x)$ is an orthonormal scaling function, and when $\phi(x)$ is the scaling function in a biorthogonal system, it satisfies the assumptions of the above theorem. Thus, in these cases, we have an elevated scaling function $\Phi(x)$ that generates a new multiresolution analysis of $L^2(\mathbb{R})$. Since in general $\sum h_k h_{k+2m} \neq 0$, the elevated $\Phi(x)$ will not be an orthonormal scaling function (we will give a proof later). We will construct a biorthogonal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$ by following the guideline of [5].

The frequency response functions for the biorthogonal system must satisfy (2.2). By (3.4), we choose

$$m_0^{\tilde{\Phi}}(\omega) = \begin{cases} \dfrac{2}{1 + e^{i\omega}} m_0^{\phi}(\omega) & \text{if } \omega \neq (2n+1)\pi, \\ 0 & \text{otherwise,} \end{cases} \tag{3.9}$$

and

$$m_1^{\tilde{\Phi}}(\omega) = -e^{-i\omega}\overline{m_0^{\Phi}(\omega + \pi)}, \qquad m_1^{\Phi}(\omega) = -e^{-i\omega}\overline{m_0^{\tilde{\Phi}}(\omega + \pi)}.$$

With these choices, the wavelet coefficients $(g_k)$ and $(\tilde{g}_k)$ are

$$g_k = (-1)^k \tilde{h}_{1-k}, \quad \tilde{g}_n = (-1)^k h_{1-k}.$$

That is,

$$\Psi(x) = \sum g_k \Phi(2x - k), \quad \tilde{\Psi}(x) = \sum \tilde{g}_k \tilde{\Phi}(2x - k).$$

Here $\tilde{h}_k$ are the coefficients in

$$m_0^{\tilde{\Phi}}(\omega) = \frac{1}{2} \sum_k \tilde{h}_k e^{-ik\omega}.$$

They are nonzero only for $k = 0, \ldots, 2p - 2$, and they satisfy

$$\tilde{h}_k = 2c_k - \tilde{h}_{k-1}. \tag{3.10}$$

And it can be shown that

$$\sum_{k=0}^{2p-2} (-1)^k k^m \tilde{h}_k = 0, \text{ where } 0 \le m \le p - 2. \tag{3.11}$$

For these choices, and when $\phi$ and $\psi$ are orthonormal, we will see that (1) the relation between $\tilde{\Phi}$ and $\tilde{\phi}$ is "reciprocal" to the relation between $\Phi$ and $\phi$; (2) $\Psi'(x-k)$ are orthogonal to each other and to $\Phi'(x-k)$; (3) $\Phi(x-k)$ are *not* orthogonal.

PROPOSITION 3.4. *With all assumptions made above,*

$$\chi_{[-1,0]} * \tilde{\Phi} = \phi. \tag{3.12}$$

The proof is done by simply taking the Fourier transform on both sides. Essentially, $m_0^{\Phi}(\omega)$ is constructed by multiplying a factor of $(1+e^{i\omega})/2$ into $m_0^{\phi}(\omega)$, and the dual $m_0^{\tilde{\Phi}}(\omega)$ is constructed by dividing a factor of $(1+e^{i\omega})/2$ from $m_0^{\phi}(\omega)$. For this reason, $\Phi$ will have higher regularity than that of $\phi$, and $\tilde{\Phi}$ will have lower regularity than that of $\phi$. This phenomenon has been observed on the compactly supported biorthogonal wavelets in [5], see next section for more discussion.

PROPOSITION 3.5. *With all assumptions made above,*

$$\Psi(x) = 4\int_{-\infty}^{x} \psi(t)\,dt \quad \forall x \in \mathbb{R}. \tag{3.13}$$

Proof: Since

$$m_1^{\Phi}(\omega) = e^{-i\frac{\omega}{2}}\left(\frac{1+e^{-i(\omega+\pi)}}{2}\right)^{-1}\overline{m_0(\omega+\pi)} = \left(\frac{1+e^{-i(\omega+\pi)}}{2}\right)^{-1} m_1^{\phi}(\omega)$$

and

$$\begin{aligned}
i\omega\hat{\Psi}(\omega) &= i\omega\, m_1^{\phi}(\frac{\omega}{2})\frac{2}{1+e^{-i(\frac{\omega}{2}+\pi)}}\hat{\Phi}(\frac{\omega}{2})\\
&= i\omega\, m_1^{\phi}(\frac{\omega}{2})\frac{2}{1-e^{-i\frac{\omega}{2}}}\frac{2(1-e^{-i\frac{\omega}{2}})}{i\omega}\hat{\phi}(\frac{\omega}{2})\\
&= 4\, m_1^{\phi}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2})\\
&= 4\,\hat{\psi}(\omega).
\end{aligned}$$

By the identity $\widehat{\Psi'}(\omega) = i\omega\hat{\Psi}(\omega)$, we have the result. □

By (3.1), we see that $\langle \Phi'(x), \Phi'(x-k)\rangle$ have values 0 except when $k=0$, and $\pm 1$, which have values 2 and $-1$ respectively. By (3.13), we see that

$$\langle \Psi'(x), \Psi'(x-k)\rangle = 16\delta_{0k}, \quad \langle \Psi'(x), \Phi'(x-k)\rangle = 0.$$

Since $\Phi$ and $\Psi$ are compactly supported (of length $2p$ and $2p-1$ respectively), the inner products $\langle \Phi(x), \Phi(x-k)\rangle$, $\langle \Phi(x), \Psi(x-k)\rangle$ and $\langle \Psi(x), \Psi(x-k)\rangle$ are nonzero only for a small number of $k$. These are the desirable properties when $\Phi$ and $\Psi$ are applied to the numerical solution of differential equations. And we note that, when $p=1$, $\Phi$ and $\Psi$ corresponds to the piecewise linear hierarchical finite element basis, as already noted in [17].

For the non-orthogonality of $\Phi(x-k)$, we first introduce a result by Yen [18].

LEMMA 3.3. *Let $\phi(x)$ be an orthonormal scaling function and $\psi(x)$ be the associated orthonormal wavelet. If both $\phi$ and $\psi$ are differentiable, then*

$$\langle \psi'(x), \psi'(x-k)\rangle = 4\langle \phi'(x), \phi'(x-2k)\rangle - \langle \phi'(x), \phi'(x-k)\rangle.$$

By this lemma, if $\Phi(x-k)$ were orthonormal and an associated orthonormal wavelet $\Psi(x)$ could be found, then $\Phi = \tilde{\Phi}$ and $\Psi = \tilde{\Psi}$ and (3.13) would still be true. So we have

$$0 = \frac{1}{16}\langle \Psi'(x), \Psi'(x-1)\rangle = \frac{1}{4}\langle \Phi'(x), \Phi'(x-2)\rangle - \frac{1}{16}\langle \Phi'(x), \Phi'(x-1)\rangle = \frac{1}{16},$$

which is absurd. So $\Phi(x-k)$ cannot be orthonormal, and a biorthogonal system is necessary.

At last, let us have a few words on the generalization to biorthogonal system. Namely, if there is a biorthogonal system $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ and we define the elevation $\Phi$ by (3.1), then all properties (3.2) through (3.6) are still true (with a minor modification on the meaning of index $p$), and $\Phi(x)$ generates another multiresolution analysis of $L^2(\mathbb{R})$. In (3.9), $\phi$ would be changed to $\tilde{\phi}$ and the the new biorthogonal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$ is constructed by the same procedure. Consequently, in (3.10), $c_k$ would be changed to $\tilde{c}_k$, and (3.11) would still be true. The deconvolution relation (3.12) holds for $\phi$ replaced by $\tilde{\phi}$. The antiderivative relation (3.13) still holds but naturally $\{\Psi'(x-k)\}$ will not be an orthogonal set.

We think $\Phi$, $\Psi$ as the elevation of $\phi$ and $\psi$, Because their smoothness and degree of approximation are *raised*. Then in this sense $\tilde{\Phi}$ and $\tilde{\Psi}$ are *lowered* from $\phi$ and $\psi$. Because of pair of raising and lowering, we call this scheme the *leverage* scheme.

**4. Examples.** Let us start with the most fundamental case of orthonormal scaling function $\phi_{(1)}(x) = \chi_{[0,1]}$ and the associated orthonormal wavelet (the Haar function) $\psi_{(1)}(x)$. $\phi_{(1)}$ has the approximation degree $p = 1$. The coefficients are

$$c_k = \{1, 1\}, \quad h_k = \{\frac{1}{2}, 1, \frac{1}{2}\}, \quad \tilde{h}_k = \{2\}.$$

See Figures 1 and 2 for $\phi_{(1)}$ and $\psi_{(1)}$ respectively. The leveraged $\Phi_{(1)}$ is the standard piecewise linear finite element basis function (the "hat" function, or the B-spline of order 2), see Figure 3. The associated $\Psi_{(1)}(x) = 2\Phi_{(1)}(2x)$ is the scaled "hat" function, see Figure 4. Observe that $\Phi_{(1)}$ and $\Psi_{(1)}$ are essentially the piecewise linear hierarchical finite element basis functions. In this case, $\tilde{\Phi}_{(1)}$ and $\tilde{\Psi}_{(1)}$ are $\delta$-functions, and we don't have a theory in $L^2(\mathbb{R})$.

In case of $p = 2$, we have the coefficients

$$c_k = \{\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\},$$

$$h_k = \{\frac{1+\sqrt{3}}{8}, \frac{2+\sqrt{3}}{4}, \frac{3}{4}, \frac{2-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{8}\},$$

$$\tilde{h}_k = \{\frac{1+\sqrt{3}}{2}, 1, \frac{1-\sqrt{3}}{2}\}.$$

See Figures 5 through 8 for $\phi_{(2)}$, $\psi_{(2)}$, $\Phi_{(2)}$ and $\Psi_{(2)}$, respectively.

The scaling coefficients $\tilde{h}_k = \{\frac{1+\sqrt{3}}{2}, 1, \frac{1-\sqrt{3}}{2}\}$ are very interesting. Among all of the lowpass filters of the form

$$(\frac{1+\sqrt{\alpha}}{2}, \quad 1, \quad \frac{1-\sqrt{\alpha}}{2}),$$

there is always a corresponding scaling function $\phi_\alpha(x)$, and $\phi_\alpha(x-k)$ can always span the constant 1. But $\alpha = 0$ is the only case that can span the linear function $x$, which is the case of B-spline of order 2 (See Figure 3). If $\alpha < 3$, $\phi_\alpha(x) \in L^2(\mathbb{R})$ and otherwise $\phi_\alpha$ are distributions, see [16]. The coefficients $\tilde{h}_k$ listed above corresponds to the case of $\alpha = 3$.

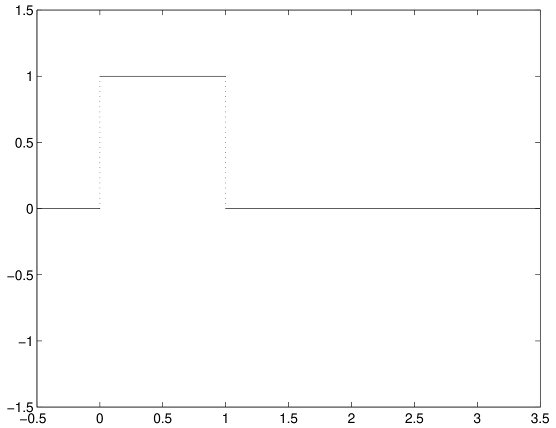


**Fig. 1** $\phi_{(1)}, \phi_{(1,1)}, \phi_{(1,3)}$

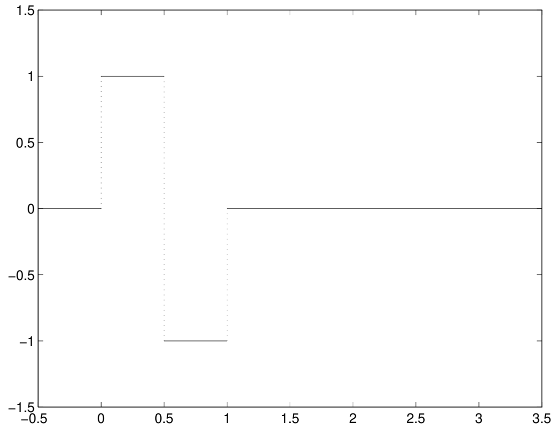


**Fig. 2** $\psi_{(1)}, \psi_{(1,1)}$

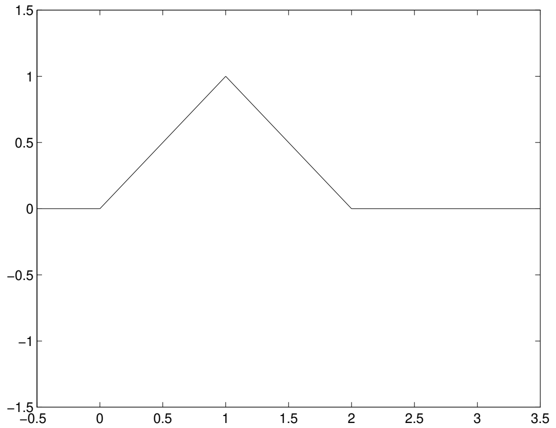


**Fig. 3** $\Phi_{(1)}, \phi_{(2,2)}, \Phi_{(1,3)}$

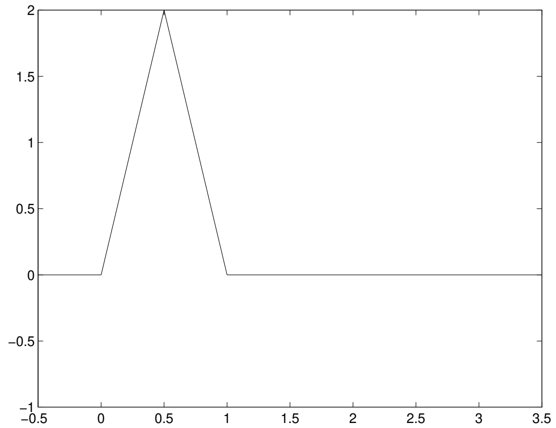


**Fig. 4** $\Psi_{(1)}$

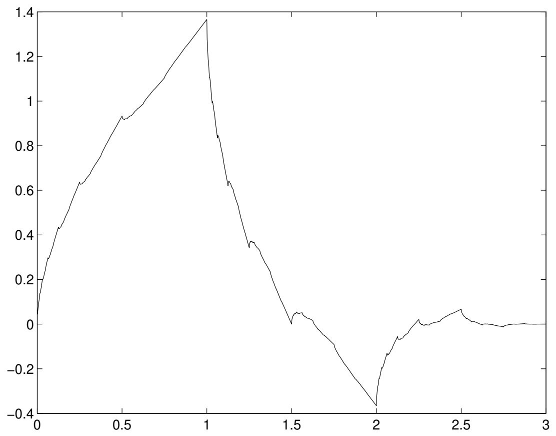


**Fig. 5** $\phi_{(2)}$

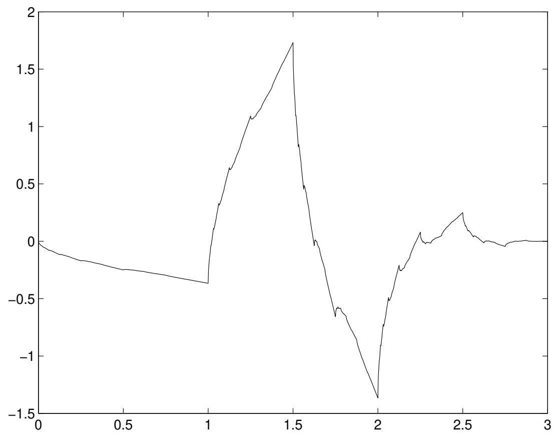


**Fig. 6** $\psi_{(2)}$

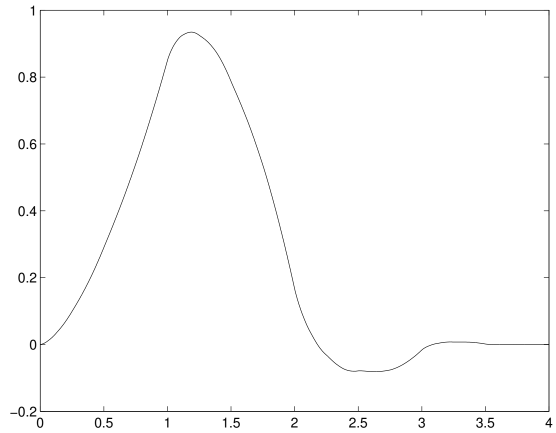


**Fig. 7** $\Phi_{(2)}$

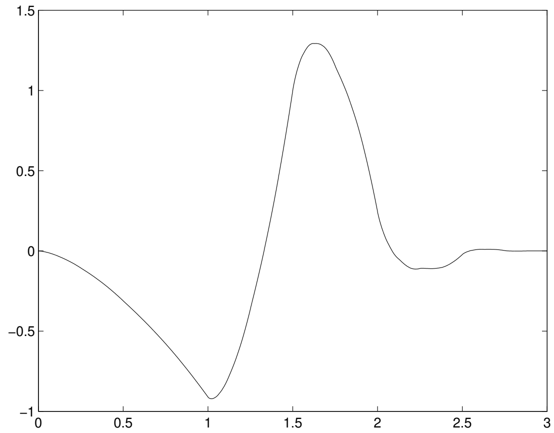


**Fig. 8** $\Psi_{(2)}$

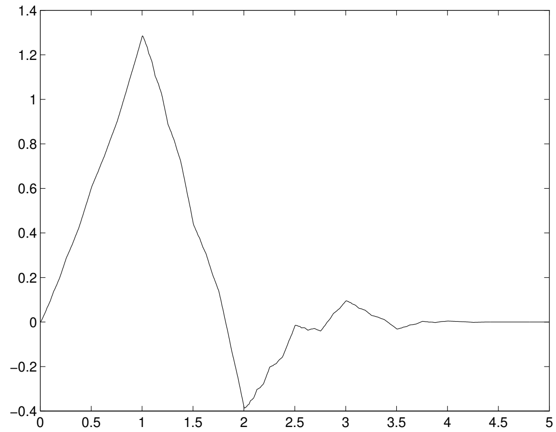


**Fig. 9** $\phi_{(3)}$

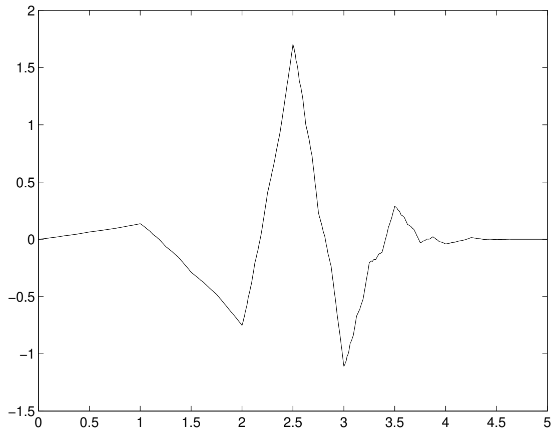


**Fig. 10** $\psi_{(3)}$

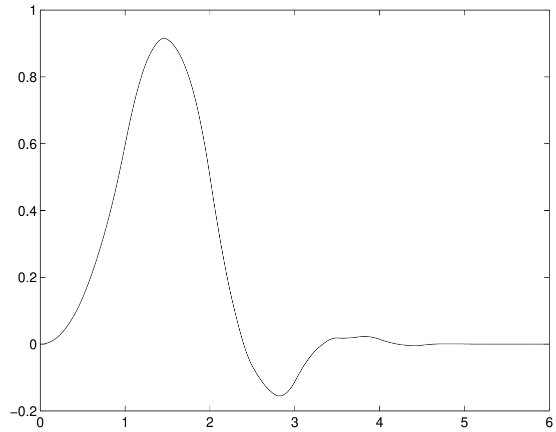


**Fig. 11** $\Phi_{(3)}$

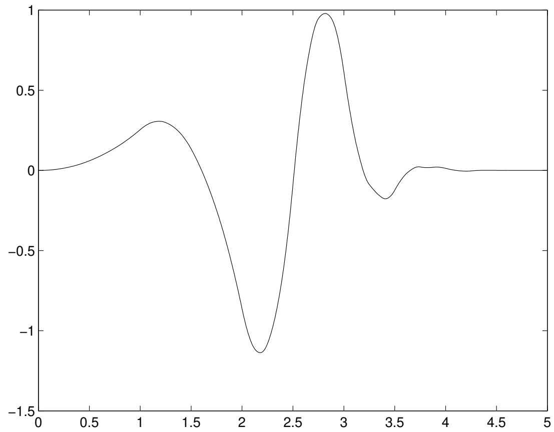


**Fig. 12** $\Psi_{(3)}$

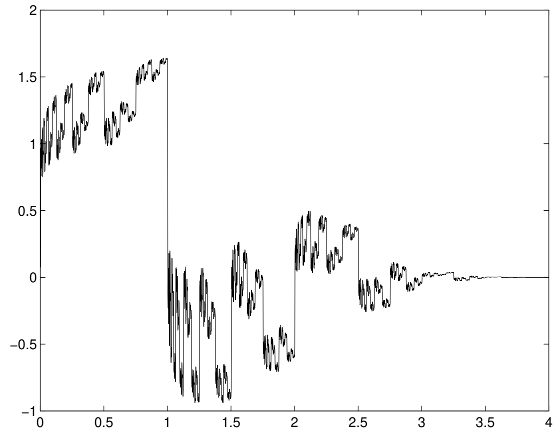

**Fig. 13** $\tilde{\Phi}_{(3)}$

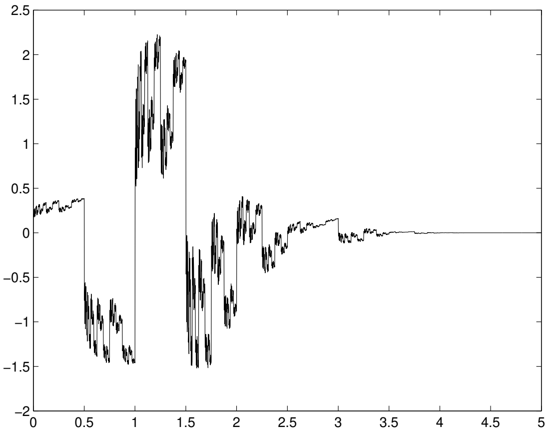

**Fig. 14** $\tilde{\Psi}_{(3)}$

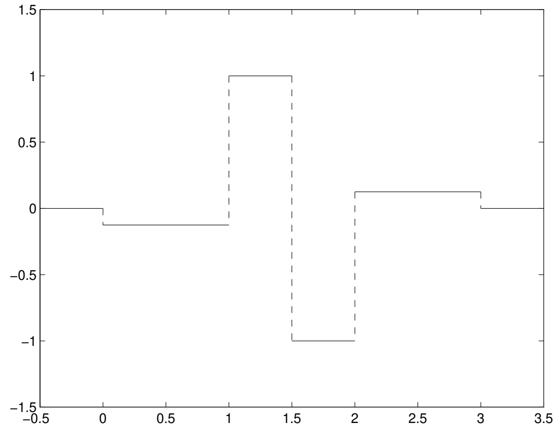

**Fig. 15** $\psi_{(1,3)}$

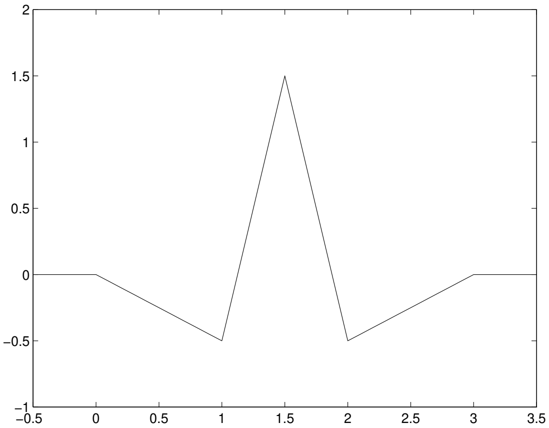

**Fig. 16** $\psi_{(2,2)}, \Psi_{(1,3)}$

In this case, supp$\tilde{\Phi}_{(2)} = [0,2]$ (in general, the length of support is the number of nonzero scaling coefficients less one). Although $\tilde{\Phi}_{(2)}(x)$ is not an $L^2$-function, we can determine its pointwise value. Let $x$ be a real number in $[0,1]$ with the binary expansion

$$x = \sum_{n\in\mathbb{N}} \frac{a_n}{2^n} \quad a_n = 0 \text{ or } 1,$$

and $\tau_n$ is defined by

$$\tau_n = \sum_{\ell=1}^{n} a_\ell - 1.$$

Let $\alpha = \frac{1+\sqrt{3}}{2}$ and $\beta = \frac{1-\sqrt{3}}{2}$. The explicit form of $\tilde{\Phi}_{(2)}$ is given by

$$\tilde{\Phi}_{(2)}(x) = \begin{cases} \sum\limits_{n\in\mathbb{N}} a_n \alpha^{n-\tau_n} \beta^{\tau_n} & \text{for } x \in (0,1] \\ 1 - \tilde{\Phi}_{(2)}(x-1) & \text{for } x \in (1,2] \\ 0 & \text{otherwise.} \end{cases}$$

For case of $p = 3$, see the orthonormal scaling function $\phi_{(3)}$ in Figure 9 and the orthonormal wavelet $\psi_{(3)}$ in Figure 10. The leveraged functions $\Phi_{(3)}$ and $\Psi_{(3)}$ are in Figures 11 and 12 respectively. In this case the dual functions $\tilde{\Phi}_{(3)}$ and $\tilde{\Psi}_{(3)}$ are in $L^2$, but very rough. See Figures 13 and 14.

As we have noted at the end of Section 3, the leverage scheme can also be applied to biorthogonal systems. Let us discuss the compactly supported biorthogonal systems by Cohen, Daubechies and Feauveau, and we refer the readers to the original paper [5] for the notations. If $\phi_{(N,\tilde{N})}(x)$ is the scaling function in a biorthogonal system, then the leveraged scaling function will be exactly $\phi_{(N+1,\tilde{N}-1)}(x)$. It can be observed from the pair of scaling coefficients $c_k$ and $\tilde{c}_k$.

For instance, in the case of $(1,3)$,*

$$c_k = \{1, 1\},$$
$$\tilde{c}_k = \{-\frac{1}{8}, \frac{1}{8}, 1, 1, \frac{1}{8}, -\frac{1}{8}\}.$$

By (3.2) and (3.10), we have

$$h_k = \{\frac{1}{2}, 1, \frac{1}{2}\}$$
$$\tilde{h}_k = \{-\frac{1}{4}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{4}\}.$$

These are exactly the coefficients for the case of $(2,2)$. $\phi_{(1,3)}$ is still $\chi_{[0,1]}$ (Figure 1), and $\Phi_{(1,3)} = \phi_{(2,2)}$ is the hat function (Figure 3). Check out the pictures of $\psi_{(1,3)}$ and $\Psi_{(1,3)} = \psi_{(2,2)}$ in Figures 15 and 16 respectively.

The most fundamental case of the biorthogonal systems is that of $(N,\tilde{N}) = (1,1)$. In this case, it is actually an orthonormal system, and $\phi_{(1,1)} = \phi_{(1)}$ and $\psi_{(1,1)} = \psi_{(1)}$. The lifting scheme of Sweldens [15] can construct the coefficients $c_k$ and $\tilde{c}_k$ from the case of $(N,\tilde{N})$ to $(N,\tilde{N}+2)$. For instance, from $(1,1)$ to $(1,3)$ or from $(2,2)$ to $(2,4)$, and so on. If we apply the leveraged scheme together with the lifting scheme, we can "lift" the coefficients from $(1,1)$ to $(1,3)$, and then "leverage" to $(2,2)$, then "lift" to $(2,4)$, and then "leverage" to $(3,3)$, and so on. In this manner, we can construct all of these compactly supported biorthogonal wavelets from the most fundamental one, that is, the characteristic function and the Haar function.

**5. Numerical Tests.** We shall start with the most elementary two-point boundary value problem $-u''(x) = 1$ for $x \in (0,1)$ with periodic boundary conditions. Let $\phi$ be the orthonormal scaling function of approximation degree $p = 2$, and $\Phi$ be the elevation of $\phi$. Let $N$ be the number of unknowns, that is, the dimension of the Galerkin-wavelets space. If we solve the problem in $V_j$, then $N$ will be approximately $2^j$. It is well known that the condition number of the stiffness matrix will be proportional to $N^2$. The similar situation happens when we build the stiffness matrix by $\Phi$. See the following table.

| sizes | $64*64$ | $128*128$ | $256*256$ | $512*512$ | $1024*1024$ |
|---|---|---|---|---|---|
| conditions | 1466 | 8287 | 46870 | 265125 | 1499760 |

Let $\psi$ be the orthonormal wavelet associated with $\phi$, and $\Psi$ be the elevation of $\psi$. If we change the basis from $\Phi$ to $\Psi$, we observe the reducing of condition numbers in the next table.

| steps | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| conditions | 1499760 | 265125 | 46870 | 11719 | 2932 |

In the table, "steps" mean the number of levels that we convert from $\Phi$ into $\Psi$. For instance, after two steps, we have changed the basis from those of $V_{10}$ to those of $W_9 \oplus W_8 \oplus V_8$.

Comparing to the configuration that uses orthonormal wavelets as the Galerkin basis, we do not need an extra preconditioner. If we use the orthonormal scaling function $\phi$ as Galerkin basis,

* Note that we have scaled the coefficients such that $\sum c_k = 2$.

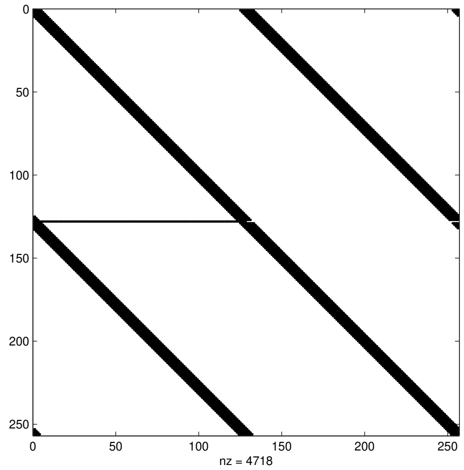


**Fig. 17** $WAW^T$

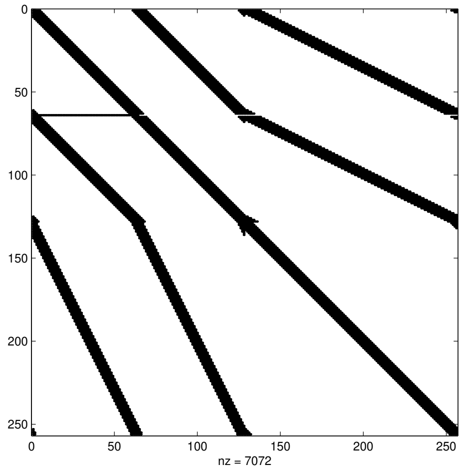


**Fig. 18** $W^2AW^{2T}$

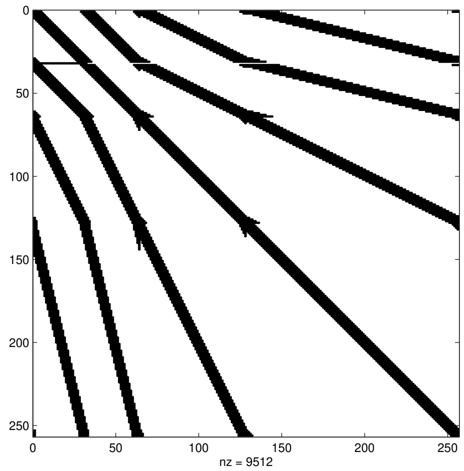


**Fig. 19** $W^3AW^{3T}$

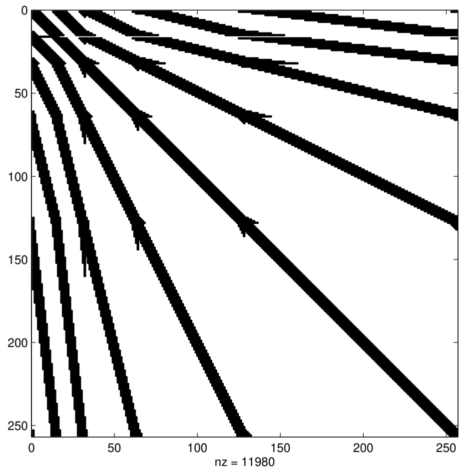


**Fig. 20** $W^4AW^{4T}$

the condition numbers of the stiffness matrices have the scale of $O(N^2)$. The following numerical example was done Daubechies orthonormal scaling functions with approximation degree $p = 3$.

| sizes | $32 * 32$ | $64 * 64$ | $128 * 128$ | $256 * 256$ | $512 * 512$ |
|---|---|---|---|---|---|
| conditions | 958 | 3840 | 15366 | 61471 | 245894 |

In this situation, the condition number does not change its scale when we change the basis from scaling to wavelet functions. In the following numerical example, we take the stiffness matrix of size $512 \times 512$; that is, let the Galerkin-wavelet space be $V_9$. We start with the basis of $V_9$. Then we change the basis to $W_8 \oplus V_8$ (1 step), and to $W_8 \oplus W_7 \oplus V_7$ (2 steps) and so on.

| steps | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| conditions | 245894 | 282646 | 291162 | 290909 | 291483 |

There are several heuristic reasons for this phenomenon, please also check the discussions in Xu and Shann [17].

The images of the stiffness matrix and its wavelet transforms are well-known in the community. But we still choose to show four pictures in case the reader is interested to see. In

Figures 17–20, the matrix size is $256 \times 256$. Figure 17 is the image of stiffness matrix with basis from $W_7 \oplus V_7$. We select a threshold, the element is painted black if its absolute value is larger than the threshold, otherwise painted white. The black portions near the corners are due to the periodization of the wavelets, and the horizontal black rule is due to the normalization condition $\int u\,dx = 1$ that ensures the uniqueness of the solution for periodic boundary conditions. In the images we see clearly that the larger entries of the transformed matrix are concentrated along the diagonals in several blocks of submatrices.

The wavelet approximation theorem estimates the scale of the diagonal elements in each block of the transformed matrix. Beylkin and his colleagues found a simple diagonal preconditioner that will lower the condition number of the stiffness matrix under wavelet basis. The key idea of this preconditioner is to balance the scale of each block within the matrix. It works fine for stiffness matrices. The following numerical example is done by repeating the previous experiment with the preconditioners.

| steps (with pre.d.) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| conditions | 245894 | 93051 | 42317 | 24366 | 18860 |

However, the preconditioner does not work for the mass matrices. To see it simply, just consider the equation $-u'' + 1 = f$. Take the orthonormal scaling functions, the mass matrix is $I$. After the preconditioning it is $D^{-1} I\, D^{-1} = D^{-2}$. By the design of $D$, $\kappa(D^{-2}) = 2^{\ell}$ where $\ell$ is proportional to the steps of wavelet transform.

Let $\phi$ be the scaling function of approximation degree $p = 2$. The linear combination of $\phi(2^j x - k)$ can reproduce polynomials of degree 1 in any finite interval. Hence the Galerkin-wavelet method using this set of basis should be comparible to that using piecewise linear polynomials. In terms of convergence rate, our numerical experiments confirm the theoretical prediction. The following numerical experiments are done with Daubechies orthonormal scaling functions of degree $p = 2$.

| N | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|
| Error | $3.9 \times 10^{-4}$ | $4.1 \times 10^{-5}$ | $8.3 \times 10^{-6}$ | $3.4 \times 10^{-6}$ | $1.6 \times 10^{-6}$ |

The convergence rate slows down slightly when the degree of freedom becomes large. It is probabely because that we did not empoly an accurate numerical quadrature scheme.

If $\Phi$ is elevated from $\phi$ of degree $p = 2$, then $\Phi$ shall be able to reproduce polynomials of degree 2 in any finite interval. And thus the Galerkin-wavelet method using this set of basis should be comparible to that using piecewise quadratic polynomials.

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