THE LEVERING SCHEME OF BIORTHOGONAL WAVELETS

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ABSTRACT

We present a scheme that will lever orthonormal or biorthogonal wavelets to a new system of biorthogonal wavelets. If we start with orthonormal wavelets, the raised scaling functions and wavelets are compactly supported and are differentiable. The derivatives of the raised biorthogonal scaling/wavelets forms an almost orthonormal system. If we start with B-splines and cooperating with the lifting scheme of Sweldens, our levering scheme can reproduce all of those biorthogonal wavelets of compact support by Cohen, Daubechies and Feauveau. There is a simple algorithm to construct from the old filter coefficients to the new filter coefficients.

1. MOTIVATION

The motivation of this work is the following formula

$$\Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt \tag{1}$$

that was suggested by Xu and Shann in their paper on Galerkin-wavelets methods [7]. Here $\phi(x)$ is an orthonormal scaling function. In that paper, they were studying the Galerkin basis functions derived from orthonormal wavelets that will produce well-conditioned stiffness matrices and will satisfy the Dirichlet boundary conditions. The formula (1) was suggested to ensure the compact supportedness of the basis functions. We follow this hint and discovered that it is actually the starting point of a general scheme to derive from a wavelet to another wavelet with some new properties. We call it the *levering scheme* of biorthogonal wavelets.

The idea that to integrate the orthonormal wavelets before applying them to the numerical solution of differential equations has been explored in other articles. For instance, an earlier work of Jawerth and Sweldens [3] has exactly the same approach as ours. And more recently we found that Ponenti and Liandrat [2] have done a good job in this approach.

Although the goal of this scheme was to the solution of differential equations, it is very interesting to realize that the levering scheme and the *lifting scheme* of Sweldens [5] can work together to re-produce all of the compactly supported biorthogonal wavelets by Cohen, Daubechies and Feauveau [1].

This article is an abridged version of our technical report [4].

2. THE LEVERING SCHEME

We assume that the readers are familiar with common notations in the theory of wavelets. Actually, we adopt almost the same notation from [1].

To have a quick start, let $\phi(x)$ be a scaling function which satisfies $\phi(x) = \sum c_k \phi(2x-k)$. We normalize $\sum c_k = 2$ and $\hat{\phi}(0) = 1$. We say that $\phi(x)$ has an approximation degree of p if linear combinations of $\phi(x-k)$ can (locally) express polynomials of degree 0 through $p-1$. Here $p \geq 1$ is an integer. Furthermore, $\phi(2^j x - k)$ generate a Multiresolution Analysis (MRA) $\{V_j\}$ of L^2 and $\{\phi(x-k)\}$ forms a Riesz basis of V_0.

We shall introduce the levering scheme by compactly supported orthonormal wavelets. That is, let $\phi(x)$ be a compactly supported orthonormal scaling function of approximation degree p, and let $\psi(x)$ be the orthonormal wavelet associated with $\phi(x)$. Let $\Phi(x)$ be the function defined in (1), it is the scaling function raised from $\phi(x)$.

It is easy to check that $\Phi(x)$ satisfies a scaling equation

$$\Phi(x) = \sum_{k=0}^{2p} h_k \Phi(2x-k)$$

with $2p+1$ nonzero scaling coefficients $h_0, h_1, \ldots, h_{2p}$.

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Here

$$h_k = \frac{(c_{k-1} + c_k)}{2}, \tag{2}$$

note that $c_{-1} = c_{2p} = 0$. Furthermore, note that $\mathrm{supp}\Phi(x) = [0, 2p]$ and $\sum h_k = \sum c_k = 2$.

Since

$$\Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt = \int_{x-1}^{x} \phi(t)\, dt,$$

we have

$$\Phi(x) = \chi_{[0,1]} * \phi(x). \tag{3}$$

Here $*$ stands for the (continuous) convolution operator. Hence

$$\hat{\Phi}(\omega) = \widehat{\chi_{[0,1]}}(\omega)\hat{\phi}(\omega) = e^{-i\frac{\omega}{2}} \frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\, \hat{\phi}(\omega).$$

Note that $\hat{\Phi}(0) = \int \Phi\, dx = 1$ and $\hat{\Phi}(2k\pi) = 0$ for all integers $k \neq 0$.

The frequency response function for $\Phi(x)$ shall be defined by $m_0^{\Phi}(\omega) = \frac{1}{2}\sum h_k e^{-ik\omega}$. And we also have

$$m_0^{\Phi}(\omega) = (\frac{1}{2} + \frac{1}{2}e^{-i\omega})m_0^{\phi}(\omega).$$

Therefore,

$$(h_n) = (\frac{1}{2}, \frac{1}{2}) * (c_n).$$

Here $*$ stands for the (discrete) convolution operator.

It is easy to verify that

$$\sum_{k=0}^{2p} (-1)^k k^m h_k = 0$$

for $0 \leq m \leq p$ and

$$\sum_k \Phi(x-k) = 1.$$

That is, $\Phi(x)$ is also a scaling function and its approximation degree is $p+1$. We can also prove that if $\phi \in L^1 \cap L^2$ is a scaling function and $\{\phi(x-k)\}$ is a Riesz basis, then $\{\Phi(x-k)\}$ is also a Riesz basis and $\Phi(2^j x - k)$ generate another multiresolution analysis of L^2. Since in general $\sum h_k h_{k+2m} \neq 0$, the raised $\Phi(x)$ will not be an orthonormal scaling function. Following the guideline of [1], we can construct a biorthogonal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$.

We choose the frequency response function for $\tilde{\Phi}(x)$ to be

$$m_0^{\tilde{\Phi}}(\omega) = \begin{cases} \dfrac{2}{1+e^{i\omega}} m_0^{\phi}(\omega) & \text{if } \omega \neq (2n+1)\pi, \\ 0 & \text{otherwise.} \end{cases}$$

The high pass filters can be constructed by

$$m_1^{\tilde{\Phi}}(\omega) = -e^{-i\omega}\overline{m_0^{\Phi}(\omega+\pi)}$$

and

$$m_1^{\Phi}(\omega) = -e^{-i\omega}\overline{m_0^{\tilde{\Phi}}(\omega+\pi)}.$$

With these choices, the wavelet coefficients g_k and $\tilde{g}_k$ are

$$g_k = (-1)^k \tilde{h}_{1-k}, \quad \tilde{g}_n = (-1)^k h_{1-k}.$$

The associated biorthogonal wavelets are

$$\begin{aligned} \Psi(x) &= \sum g_k \Phi(2x-k), \\ \tilde{\Psi}(x) &= \sum \tilde{g}_k \tilde{\Phi}(2x-k). \end{aligned}$$

Here $\tilde{h}_k$ are the coefficients in

$$m_0^{\tilde{\Phi}}(\omega) = \frac{1}{2}\sum_k \tilde{h}_k e^{-ik\omega}.$$

They are nonzero only for $k = 0, \ldots, 2p-2$, and they satisfy

$$\tilde{h}_k = 2c_k - \tilde{h}_{k-1}. \tag{4}$$

With all the construction made above, we have

$$\chi_{[-1,0]} * \tilde{\Phi} = \phi \tag{5}$$

and

$$\sum_{k=0}^{2p-2} (-1)^k k^m \tilde{h}_k = 0,$$

for $0 \leq m \leq p-2$. That is, $\tilde{\Phi}$ has an approximation degree of $p-1$.

In the sense of (3) and (5), and in the sense that the approximation degrees of Φ and $\tilde{\Phi}$ are respectively $p+1$ and $p-1$, we see that the levering scheme has *raised* Φ from ϕ and *lowered* $\tilde{\Phi}$ from ϕ.

The raised wavelet Ψ will be a primitive of ψ. One can verify that

$$\Psi(x) = 4\int_{-\infty}^{x} \psi(t)\, dt. \tag{6}$$

So $\{\Psi'(x-k)\}$ is an orthogonal set and they are orthogonal to $\Phi'(x-k)$. Note that, by the construction of Φ, $\Phi'(x)$ will be orthogonal to $\Phi'(x-k)$ for $|k| > 1$.

The entire procedure of construction can apply to a biorthogonal system $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$. Essentially everything will follow except that $\Psi'(x-k)$ will not be an orthogonal system.

3. EXAMPLES

Let us start with the most fundamental case of orthonormal scaling function $\phi_{(1)}(x) = \chi_{[0,1]}$ and the Haar wavelet $\psi_{(1)}(x)$. $\phi_{(1)}$ has the approximation degree $p = 1$. The coefficients are

$$c_k = \{1, 1\}, \quad h_k = \{\frac{1}{2}, 1, \frac{1}{2}\}, \quad \tilde{h}_k = \{2\}.$$

The raised $\Phi_{(1)}$ is the standard piecewise linear finite element basis function (the "hat" function, or the B-spline of order 2). The associated wavelet $\Psi_{(1)}(x) = 2\Phi_{(1)}(2x)$, it is a scaled "hat" function. Observe that $\Phi_{(1)}$ and $\Psi_{(1)}$ are essentially the one-dimensional piecewise linear hierarchical finite element basis functions. In this case, $\tilde{\Phi}_{(1)}$ and $\tilde{\Psi}_{(1)}$ are δ-functions, and we do not have a system in L^2.

In case of $p = 2$, we have the coefficients

$$\begin{aligned} c_k &= \{\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\}, \\ h_k &= \{\frac{1+\sqrt{3}}{8}, \frac{2+\sqrt{3}}{4}, \frac{3}{4}, \frac{2-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{8}\}, \\ \tilde{h}_k &= \{\frac{1+\sqrt{3}}{2}, 1, \frac{1-\sqrt{3}}{2}\}. \end{aligned}$$

The dual scaling coefficients $\tilde{h}_k = \{\frac{1+\sqrt{3}}{2}, 1, \frac{1-\sqrt{3}}{2}\}$ are very interesting. Among all length-3 low-pass filters of the form

$$(\frac{1+\sqrt{\alpha}}{2}, \quad 1, \quad \frac{1-\sqrt{\alpha}}{2}),$$

there is always a corresponding scaling function $\phi_\alpha(x)$, and $\phi_\alpha(x-k)$ can always span the constant 1. But $\alpha = 0$ is the only case that can span the linear function x, which is the case of B-spline of order 2. If $\alpha < 3$ then $\phi_\alpha(x) \in L^2$, otherwise ϕ_α is a distribution, see [6]. The coefficients $\tilde{h}_k$ listed above corresponds to the case of $\alpha = 3$.

Although $\tilde{\Phi}_{(2)}(x)$ is not an L^2-function, we can determine its values at all dyadic points. Let x be a real number in $[0, 1]$ with the binary expansion

$$x = \sum \frac{a_n}{2^n}$$

where a_n is either 0 or 1, and τ_n is defined by

$$\tau_n = \sum_{\ell=1}^{n} a_\ell - 1.$$

Let $\beta = \frac{1-\sqrt{3}}{2}$ and $\gamma = \frac{1+\sqrt{3}}{2}$. The explicit form of $\tilde{\Phi}_{(2)}$ is given by

$$\tilde{\Phi}_{(2)}(x) = \begin{cases} \sum a_n \beta^{\tau_n} \gamma^{n-\tau_n} & \text{for } x \in (0, 1] \\ 1 - \tilde{\Phi}_{(2)}(x-1) & \text{for } x \in (1, 2] \\ 0 & \text{otherwise.} \end{cases}$$

4. THE BIORTHOGONAL SYSTEMS

As we have noted in Section 2, the levering scheme can also be applied to biorthogonal systems. Let $\phi_{(N,\tilde{N})}$ be a compactly supported scaling function in a biorthogonal system constructed by Cohen, Daubechies and Feauveau [1]. Then the raised scaling function $\Phi_{(N,\tilde{N})}$ will be exactly $\phi_{(N+1,\tilde{N}-1)}$. This fact can be observed from the pair of scaling coefficients h_k and $\tilde{h}_k$.

For instance, in the case of $\phi_{(1,3)}$, we have

$$\begin{aligned} c_k &= \{1, 1\}, \\ \tilde{c}_k &= \{-\frac{1}{8}, \frac{1}{8}, 1, 1, \frac{1}{8}, -\frac{1}{8}\}. \end{aligned}$$

Note that we have normalized c_k to $\sum c_k = 2$. By (2) and (4), we have

$$\begin{aligned} h_k &= \{\frac{1}{2}, 1, \frac{1}{2}\} \\ \tilde{h}_k &= \{-\frac{1}{4}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{4}\}. \end{aligned}$$

These are exactly the coefficients for $\phi_{(2,2)}$. $\phi_{(1,3)}$ is still $\chi_{[0,1]}$, and $\Phi_{(1,3)} = \phi_{(2,2)}$ is the hat function. The pictures of $\psi_{(1,3)}$ and $\Psi_{(1,3)} = \psi_{(2,2)}$ are already published in many articles.

The most fundamental case of the biorthogonal systems is that of $(N, \tilde{N}) = (1, 1)$. In this case, it is actually an orthonormal system with $\phi_{(1,1)} = \phi_{(1)}$ and $\psi_{(1,1)} = \psi_{(1)}$. The *lifting scheme* of Sweldens [5] can construct the coefficients c_k and $\tilde{c}_k$ from the case of $(N, \tilde{N})$ to $(N, \tilde{N}+2)$. For instance, from $(1, 1)$ to $(1, 3)$ or from $(2, 2)$ to $(2, 4)$. If we apply the levering scheme together with the lifting scheme, we can *lift* the coefficients from $(1, 1)$ to $(1, 3)$, and then *lever* it to $(2, 2)$, then *lift* it to $(2, 4)$, and then *lever* it to $(3, 3)$, and so on. In this manner, we can construct all of these compactly supported biorthogonal wavelets from the most fundamental one; that is, the characteristic function and the Haar wavelet.

5. REFERENCES

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