MATRICES AND QUADRATURE RULES FOR WAVELETS*

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Abstract. Using the scaling equations, quadratures involving polynomials and scaling/wavelet functions can be evaluated by linear algebraic equations (which are theoretically exact) instead of numerical approximations. We study two matrices which are derived from these kinds of quadratures. These particular matrices are also seen in the literature of wavelets for other purposes.

Key words: scaling equation, wavelet, quadrature, polynomial, Neumann series

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1. Introduction. In one way or another, matrices appear in almost every category of numerical problem. Accordingly we have studied matrices in general or in various particular patterns. It is not surprising that new kinds of matrices are developed when we need numerical results from the applications of wavelets or scaling functions. In this case, the *scaling equation*

$$\phi(x) = \sum_k c_k \phi(2x-k) \tag{1.1}$$

plays a key role. A numerical problem is typically converted to linear systems of equations involving the *scaling coefficients* c_k. For instance, evaluation of $\phi(x)$ at integers becomes an eigenproblem and Mallat's pyramid algorithm can be written as a sequence of matrix-vector multiplications. See the review article of Strang [17].

Because of their similarities with Fourier bases and finite element bases, wavelets are studied as a tool in scientific computation and numerical solution of differential equations. See for instance [1, 4, 8, 9, 14, 19]. In these applications, one of the fundamental computing steps is the numerical quadrature of the forms $\int f(x)\phi(x)\,dx$ or $\int f(x)\phi(x)\phi(x-k)\,dx$. The quadratures are assumed to be on the whole real line $\mathbb{R}$ if we do not specify the limits. When non-periodic boundary conditions are assumed, the foregoing quadratures may be limited on a bounded interval; then we shall write

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$\int_a^b$ instead of $\int$. Since the derivatives of ϕ also satisfy (other) scaling equations, the techniques derived for the quadratures about ϕ can be carried to the quadratures about the derivatives of ϕ.

Since the *scaling function* $\phi(x)$ and the associated *wavelet* $\psi(x)$ have the linear relation

$$\psi(x) = \sum_k (-1)^k c_{1-k} \phi(2x-k), \tag{1.2}$$

we usually only have to deal with the quadratures involving $\phi(x)$.

We will introduce more properties of scaling coefficients and wavelets in Section 2.

Numerical quadrature rules of the form

$$\int f(x)\phi(x)\,dx \approx \sum_{i=1}^{m} \alpha_i f(\xi_i)$$

are usually based on the fact that $f(x)$ can be interpolated by polynomials; the approximation error can be estimated if f is smooth enough. For instance, see the "multi-point rules" derived by Beylkin, Coifman and Rokhlin [2] and Sweldens and Piessens [18]. Therefore, it is fundamental to evaluate the quadratures where $f(x)$ is a polynomial.

Let us define

$$M^p_{m,k} = \int x^m \phi(x-k)\,dx.$$

Here p is the order of the wavelet. That is, polynomials x^m of degree $0 \le m \le p-1$ can be spanned by $\phi(x-k)$ in any closed interval. By the change of variable and binomial expansion, we see that the core elements are $\int x^m \phi(x)\,dx$. It is now well known that they can be evaluated recursively by

$$\int x^m \phi(x)\,dx = \frac{1}{2(2^m-1)} \sum_k c_k \sum_{\ell=1}^{m} \binom{m}{\ell} k^\ell \int x^{m-\ell}\phi(x)\,dx. \tag{1.3}$$

This recursion relation starts from $\int \phi\,dx = 1$.

In Section 3 we will demonstrate how the quadratures

$$\int_a^b x^m \phi(x)\,dx \tag{1.4}$$

are related to linear systems of equations. We will also show that this family of matrices has a uniform upper bound for their ℓ^2-norms. That is, the upper bound of the norms do not change with the order of wavelets p.

Beylkin [3] showed that the values of $\int \phi'(x)\phi(x-k)\,dx$ consist a normalized eigenvector of a certain matrix. Dahmen and Micchelli [5] have general theorems which guarantee that quadratures of the form

$$\int \phi^{(s_0)}(x)\phi^{(s_1)}(x-k_1)\cdots\phi^{(s_n)}(x-k_n)\,dx$$

always associate with eigenproblems and that there are unique solutions, provided appropriate assumptions. In Section 4 we will show that the quadratures

$$\int x^m\phi(x)\phi(x-k)\,dx \tag{1.5}$$

are also associated with linear systems of equations. The associated matrices also have a uniform bound in their ℓ^2-norms.

2. Scaling coefficients and wavelets. Wavelets in this article are those discovered by Daubechies [6], which have compact supports and form an orthonormal basis of $L^2(\mathbb{R})$. In this case, the wavelet $\psi(x)$ is derived from a scaling function $\phi(x)$ which satisfies the scaling equation (1.1).

The coefficients c_k were computed by constructing a certain trigonometric polynomial $m_0(\xi) = \frac{1}{2}\sum c_k e^{-ik\xi}$, see Daubechies' "Ten Lectures" [7]. However, Strang [17] pointed out that c_n can also be solved by the following relations:

$$c_k = 0 \text{ for } k \notin \{0, 1, \ldots, 2p-1\}, \tag{2.1}$$

$$\sum_k c_k - 2 = 0, \tag{2.2}$$

$$\sum_k (-1)^k k^m c_k = 0, \quad \text{for } 0 \le m \le p-1, \quad (0^0 := 1) \tag{2.3}$$

and

$$\sum_k c_k c_{k-2m} - 2\delta_{0m} = 0. \tag{2.4}$$

We shall need the relations (2.2) and (2.4) later.

Here, p is a given positive integer. We shall call, respectively, such defined c_k, ϕ and ψ the scaling coefficients, scaling functions, and wavelets of order p. This parameter p plays several roles in the theory of wavelets. What we need to know here is that $\mathrm{supp}\phi = [0, 2p-1]$, and if $\psi(x)$ is a wavelet of order p, then $\int x^m \psi \, dx = 0$ for $0 \le m \le p-1$.

Note that, by (2.1), there are $2p$ unknowns of c_k. They are to be determined by the system of equations in (2.2)–(2.4). It is clear that (2.2) and (2.3) contribute $p+1$ equations. Actually, (2.4) contributes exactly $p-1$ equations to the system, namely $1 \le m \le p-1$. It can be seen as in the following. The situations for $m \ge p$ and $m \le -p$ are consequences from (2.1); those for $1-p \le m \le -1$ are dual to the cases of $1 \le m \le p-1$; the case of $m=0$ can be derived from (2.2), the $m=0$ case of (2.3), and the $m \ne 0$ part of (2.4). Therefore, there are exactly $2p$ equations with $2p$ unknowns.

Numerical values of c_k for $2 \le p \le 10$ are listed in a table [7, p. 195] with 16 digits.[3] Those with slightly higher precision, in which $2 \le p \le 14$, can be found in [16].

In designing the algorithms for wavelet computation, it is a general strategy to reduce the computation to the manipulation of c_k. Thus the accuracy of c_k heavily determines the accuracy of these algorithms.

3. Integrals on half lines. For one-dimensional boundary value problems, the domain of interest is usually a bounded interval. In this situation, quadratures like $\int_a^b q(x)\phi(2^j x - k)\, dx$ will appear in order. In practice, one can rescale the problem and only use those wavelets and scaling functions in such fine scales that in the support of each $\phi(2^j x - k)$ (or $\psi(2^j x - k)$) there lies at most one boundary point. Since the situations are analogous at either boundary, we now suppose $x = 0$ is the left boundary point and consider the quadratures on $[0, \infty)$.

After changing variables, the core formulae for these kinds of quadratures are

$$N^p_{m,k} = \int_0^\infty x^m \phi(x-k)\, dx.$$

[3] In [7], equation (2.2) is replaced by $\sum c_k = \sqrt{2}$. Here we follow the notation in [17].

For the translation parameters $k \le 1-2p$, the quadratures are zero. For $k \ge 0$, quadratures over $[0,\infty)$ is the same as over $\mathbb{R}$. Thus $N^p_{m,k} = M^p_{m,k}$. We now concentrate on the cases where $2-2p \le k \le -1$.

We start from (1.1) with a change of variable:

$$\int_0^\infty x^m \phi(x-k)\,dx = \frac{1}{2^{m+1}} \sum_\ell c_\ell \int_0^\infty x^m \phi(x-(2k+\ell))\,dx. \tag{3.1}$$

If $1-p \le k \le -1$, over half of the support is still inside of $[0,\infty)$. Then the supports of $\phi(x-(2k+\ell))$, for $-2k \le \ell \le 2p-1$, actually lie entirely in $[0,\infty)$. Over all, we have

$$N^p_{m,k} = \frac{1}{2^{m+1}} \Big[\sum_{\ell<-2k} c_\ell N^p_{m,2k+\ell} + \sum_{\ell \ge -2k} c_\ell M^p_{m,2k+\ell} \Big].$$

Suppose the part of $\ell \ge -2k$ is known. Then we have a linear system in $N^p_{m,k}$.

Given a $p \ge 2$ ($p=1$ is trivial) and a fixed $m \ge 0$, we have unknowns $N^p_{m,k}$ for $2-2p \le k \le -1$. Now let

$$\begin{cases} x_k = N^p_{m,-k}, & \\ & k = 1,2,\ldots,2p-2. \\ b_k = \dfrac{1}{2^{m+1}} \displaystyle\sum_{\ell=2k}^{2p-1} c_\ell M^p_{m,\ell-2k}, & \end{cases}$$

Let $\boldsymbol{b} = (b_k)^T$, and $A = (a_{k\ell})$ be the matrix of order $2p-2$ such that $a_{k\ell} = c_{2k-\ell}$. Then, $\boldsymbol{x} = (x_k)^T$ is the solution of the system

$$(I - \frac{1}{2^{m+1}} A)\,\boldsymbol{x} = \boldsymbol{b}.$$

Note that A is independent of m and it is a finite submatrix of the low pass filter matrix L defined in [17], in which it is also shown that A has an eigenvalue $\lambda = 1$ with an associated eigenvector $(\phi(1), \phi(2), \ldots, \phi(2p-2))^T$. In the following we claim another property for A.

Proposition 3.1. *Let $A = A(p)$ be the matrix defined above. $\|\cdot\|$ be the matrix norm induced by the Euclidean norm. We have*

$$\|A\| \le \sqrt{2}.$$

Proof. Since $A = \begin{pmatrix} S \\ T \end{pmatrix}$ where

$$S = \begin{pmatrix} c_1 & c_0 & & 0 \\ & & \ddots & \\ c_{2p-3} & c_{2p-4} & \cdots & c_0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} c_{2p-1} & c_{2p-2} & \cdots & & c_2 \\ & \ddots & & & \\ 0 & & & c_{2p-1} & c_{2p-2} \end{pmatrix}.$$

By (2.4), $ST^T = TS^T = 0$. Thus

$$AA^T = \begin{pmatrix} SS^T & 0 \\ 0 & TT^T \end{pmatrix}.$$

Also by (2.4), we have $SS^T + TT^T = 2I$. Thus if λ is an eigenvalue of SS^T with an associated eigenvector $\boldsymbol{v}$, then λ and $2-\lambda$ are eigenvalues of AA^T with associated eigenvectors $\begin{pmatrix} \boldsymbol{v} \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{0} \\ \boldsymbol{v} \end{pmatrix}$, respectively. Since AA^T is symmetric and positive semi-definite, we have $0 \le \lambda \le 2$. Therefore

$$\|AA^T\| \le 2. \qquad \blacksquare$$

Thus, for any $m \ge 0$ and $p \ge 2$, the *Neumann series*

$$\sum_{n=0}^{\infty} (\frac{1}{2^{m+1}} A)^n \tag{3.2}$$

is an efficient method for computing the inverse of $I - \frac{1}{2^{m+1}} A$. We have prepared values for $2 \le p \le 14$ and $0 \le m \le 14$ in [16].

As for the right boundary point, one can translate the problem to the integral on $(-\infty, 0]$. Clearly

$$\int_{-\infty}^{0} x^m \phi(x-k)\, dx = M_{m,k}^p - N_{m,k}^p.$$

4. Convolutions. Now we consider the integrals of the form

$$L_{m,k}^p = \int x^m \phi(x)\phi(x-k)\, dx.$$

These are the fundamental ingredients for the numerical quadratures

$$\int q(x)\phi(2^j x - \ell)\phi(2^k x - m)\, dx.$$

Since $L^p_{0,k} = \delta_{0k}$, we consider the cases for $m \geq 1$. The support of $\phi(x)$ of order p is $[0, 2p-1]$, hence we have unknowns $L^p_{m,k}$ for $2-2p \leq k \leq 2p-2$. Then

$$
\begin{aligned}
L^p_{m,k} &= \int x^m \sum_n c_n \phi(2x-n) \sum_\ell c_\ell \phi(2x-2k-\ell)\, dx \\
&= \tfrac{1}{2} \textstyle\sum_{n,\ell} c_n c_\ell \int (\frac{x+n}{2})^m \phi(x)\phi(x+n-2k-\ell)\, dx \qquad (4.1)\\
&= \frac{1}{2^{m+1}} \Big[\sum_{n,\ell} c_n c_\ell L^p_{m,2k+\ell-n} + \sum_{n,\ell} c_n c_\ell \sum_{r=1}^{m} \binom{m}{r} n^r L_{m-r,2k+\ell-n} \Big].
\end{aligned}
$$

The above system of linear equations can be written as

$$
(I - \frac{1}{2^{m+1}} A)\boldsymbol{x} = \boldsymbol{b}. \tag{4.2}
$$

Let $N = 2p-2$, then $\boldsymbol{x} = (L^p_{m,-N}, \ldots, L^p_{m,N})^T$, $\boldsymbol{b} = (b_k)$ and $A = (a_{kj})$ with

$$
b_k = \frac{1}{2^{m+1}} \sum_{n=1}^{2p-1} c_n \sum_{\ell=\ell_1}^{\ell_2} c_\ell \sum_{r=1}^{m} \binom{m}{r} n^r L^p_{m-r,2k+\ell-n}, \quad \text{where } \begin{cases} \ell_1 = \max\{n-2k-N, 0\}, \\ \ell_2 = \min\{n-2k+N, 2p-1\} \end{cases}
$$

and

$$
a_{kj} = \sum_n c_n c_{n+j-2k}, \qquad -N \leq j,k \leq N.
$$

This matrix A was considered by Lawton [11] as a sufficient and necessary condition on c_k for the construction of an orthonormal wavelet basis. The condition is concerned with the eigenvalue 2 of A. It is also summerized by Daubechies [7] as one of three equivalent conditions.

Note that, by (1.1), the derivatives of ϕ (if they exist) also satisfy dilation equations

$$
\phi^{(s)}(x) = \sum_k 2^s c_k \phi^{(s)}(2x-k). \tag{4.3}
$$

Since $\phi \in C^{r(p)}$ and $r(p) \to \infty$ as $p \to \infty$, we can apply (4.3) in $\int \phi(x)\phi^{(s)}(x-k)\, dx$ and repeat the steps in (4.1). We find that the matrices $A = A(p)$ will have eigenvalues 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, ..., as $p \to \infty$.

Actually A has a simple pattern. On the even-numbered columns, by (2.4), all entries are 0 except $a_{k,2k} = 2$. On the odd-numbered columns, all entries are 0 except for $\lceil \frac{j-2p}{2} \rceil \leq k \leq \lfloor \frac{j+2p}{2} \rfloor$, which is the $2p$-element vector

$$\boldsymbol{h} = \begin{pmatrix} \sum_k c_k c_{k+(2p-1)} \\ \vdots \\ \sum_k c_k c_{k+3} \\ \sum_k c_k c_{k+1} \\ \sum_k c_k c_{k+1} \\ \sum_k c_k c_{k+3} \\ \vdots \\ \sum_k c_k c_{k+(2p-1)} \end{pmatrix} = \begin{pmatrix} \int \phi(x)\phi(x-\frac{2p-1}{2})\,dx \\ \vdots \\ \int \phi(x)\phi(x-\frac{3}{2})\,dx \\ \int \phi(x)\phi(x-\frac{1}{2})\,dx \\ \int \phi(x)\phi(x-\frac{1}{2})\,dx \\ \int \phi(x)\phi(x-\frac{3}{2})\,dx \\ \vdots \\ \int \phi(x)\phi(x-\frac{2p-1}{2})\,dx \end{pmatrix}.$$

Proposition 4.1. *Let $A = A(p)$ be the matrix defined in this section, $\|\cdot\|$ be the matrix norm induced by the Euclidean norm. We have*

$$\|A\| \le 2\sqrt{2}.$$

Proof. Define matrices

$$B = \begin{pmatrix} c_0 & c_1 & & \cdots & c_{2p-1} & & & \\ & c_0 & c_1 & \cdots & c_{2p-2} & c_{2p-1} & & \\ & & \ddots & & & & & \\ & & & c_0 & c_1 & & \cdots & c_{2p-1} \end{pmatrix}_{(2N+1)\times 2(N+p)}$$

and

$$D = \begin{pmatrix} c_{2p-2} & c_{2p-1} & & & & \\ & \ddots & & & & \\ & & c_0 & c_1 & c_2 & c_3 \\ & & & & c_0 & c_1 \end{pmatrix}_{(2N+1)\times 2(N+p)}.$$

That is, $B_{ij} = c_{j-i}$ and $D_{ij} = c_{(2p-1)+j-2i}$. Observe that

$$A = D\,B^T.$$

We split D into three parts:

$$D = \begin{pmatrix} E \\ F \\ G \end{pmatrix} \begin{matrix} p-1 \\ 2p-1 \\ p-1 \end{matrix}$$

Precisely, they are

$$E_{ij} = c_{2(p-i)+j-1}$$

$$F_{ij} = c_{j-2(i-1)-1}$$

$$G_{ij} = c_{2(p-i)-1-2(N+p)+j}$$

Let us consider the normal matrix DD^T. On the diagonal, there are symmetric matrices EE^T, GG^T and FF^T. For these symmetric matrices, we consider the entries on the upper triangle, that is, $i \le j$. Now

$$\begin{aligned}(EE^T)_{ij} &= \sum_{k=1}^{2(N+p)} E_{ik}E_{jk} \\ &= \sum_{k=1}^{2i} c_{2(p-i)+k-1}c_{2(p-j)+k-1} \\ &= \sum_{k=2(p-i)}^{2p-1} c_k c_{k-2(j-i)}\end{aligned}$$

and

$$\begin{aligned}(GG^T)_{ij} &= \sum_{k=1}^{2(N+p)} G_{ik}G_{jk} \\ &= \sum_{k=2(N+p)-2(p-j)+1}^{2(N+p)} c_{2(p-i)-1-2(N+p)+k}c_{2(p-j)-1-2(N+p)+k} \\ &= \sum_{k=2(j-i)}^{2(p-i)-1} c_k c_{k-2(j-i)}\end{aligned}$$

Hence

$$(EE^T)_{ij} + (GG^T)_{ij} = \sum_{m=2(j-i)}^{2p-1} c_m c_{m-2(j-i)} = \sum_m c_m c_{m-2(j-i)}.$$

By (2.4), we have $EE^T + GG^T = 2I_{p-1}$.

Next, we have

$$\begin{aligned}(FF^T)_{ij} &= \sum_{k=1}^{2(N+p)} F_{ik}F_{jk} \\ &= \sum_{k=2j-1}^{2(i-1)+2p} c_{k-2(i-1)-1}c_{k-2(j-1)-1} \\ &= \sum_{k=2(j-i)}^{2p-1} c_k c_{k-2(j-i)}\end{aligned}$$

This means $FF^T = 2I_{2p-1}$.

Then we consider the off-diagonal submatrices of DD^T. For instance

$$(EF^T)_{ij} = \sum_{k=1}^{2(N+p)} E_{ik}F_{jk}.$$

For $2j-1>2i$, it is trivial to see that $(EF^T)_{ij}=0$. For $2j-1<2i<2(j-1)+2p-2$, we have

$$\begin{aligned}(EF^T)_{ij} &= \sum_{k=2j-1}^{2i} c_{2(p-i)+k-1}c_{k-2(j-1)-1} \\ &= \sum_{m=2(p-i+j-1)}^{2p-1} c_m c_{m-2(p-i+j-1)} \\ &= 0.\end{aligned}$$

Hence $EF^T=0$. Similarly, one can show that $GF^T=0$. It is easy to derive $EG^T=0$ by comparing the nonzero elements of the matrices E and G.

In conclusion, we have

$$DD^T = \begin{pmatrix} EE^T & & \\ & 2I & \\ & & GG^T \end{pmatrix}.$$

By the similar argument used in the proof of Proposition 3.1, if $\lambda \geq 0$ is an eigenvalue of DD^T, then either $\lambda = 2$ or $2-\lambda \geq 0$ is also an eigenvalue. Hence $\|D\| = \sqrt{2}$.

Now we turn to the norm of B. Let $\boldsymbol{x}$ be a vector of length $2(N+p)$. Extend $\boldsymbol{x}$ to $\tilde{\boldsymbol{x}} = (\tilde{x}_k) \in \ell^2$ such that

$$\tilde{x}_k = \begin{cases} x_k & \text{if } 1 \leq k \leq 2(N+p) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}\|B\boldsymbol{x}\|^2 &= \sum_{i=1}^{2N+1}\Big(\sum_j c_{j-i}\tilde{x}_j\Big)^2 \\ &= \sum_{i=1}^{2N+1}\sum_{j,k} c_{j-i}c_{k-i}\tilde{x}_j\tilde{x}_k \\ &= \sum_{i=1}^{2N+1}\sum_{j,k} c_j c_k \tilde{x}_{j+i}\tilde{x}_{k+i} \\ &= \sum_k c_k^2 \sum_{i=1}^{2N+1}\tilde{x}_{k+i}^2 + \sum_{j\neq k} c_j c_k \sum_{i=1}^{2N+1}\tilde{x}_{j+i}\tilde{x}_{k+i}.\end{aligned}$$

We have $\sum_{i=1}^{2N+1}\tilde{x}_{k+i}^2 \leq \|\tilde{\boldsymbol{x}}\|^2$ and, by Cauchy-Schwarz inequality, $\sum_{i=1}^{2N+1}\tilde{x}_{j+i}\tilde{x}_{k+i} \leq \|\tilde{\boldsymbol{x}}\|^2$. Therefore

$$\begin{aligned}\|B\boldsymbol{x}\|^2 &\leq \sum_k c_k^2\|\tilde{\boldsymbol{x}}\|^2 + \sum_{j\neq k} c_j c_k \|\tilde{\boldsymbol{x}}\|^2 \\ &= \Big(\sum_k c_k\Big)^2\|\boldsymbol{x}\|^2 \\ &= 4\,\|\boldsymbol{x}\|^2.\end{aligned}$$

Hence $\|B\| \leq 2$ and $\|A\| \leq \|D\|\,\|B\| \leq 2\sqrt{2}$. ■

REFERENCES

1 E. Bacry, S. Mallat and G. Papanicolaou, A wavelet based space-time adaptive numerical method for partial differential equations, *RAIRO Math Modelling Numer. Anal.,* 26: 793–834 (1992)

2 G. Beylkin R. Coifman and V. Rokhlin, Fast wavelet transforms and numerical algorithms, *Comm. Pure Appl. Math.,* 44: 141–183 (1991)

3 G. Beylkin, On the presentation of operators in bases of compactly supported wavelets, *SIAM J. Numer. Anal.,* 6: 1716–1740 (1992)

4 G. Beylkin, R. Coifman and V. Rokhlin, Wavelets in numerical analysis, in *Wavelets and their applications,* Jones and Barlett Publishers, Boston, 1992.

5 W. Dahmen and C. A. Micchelli, Using the refinement equation for evaluating integrals of wavelets, *SIAM J. Num. Anal.,* 30: 507–537 (1993)

6 I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.,* 41: 909–996 (1988)

7 I. Daubechies, *Ten lectures on wavelets,* SIAM, Philadelphia, 1992.

8 R. Glowinski, W. M. Lawton, M. Ravachol and E. Tenenbaum, Wavelets solution of linear and nonlinear elliptic, parabolic and hyperbolic problems in one space dimension, in *Computing methods in applied sciences and engineering,* SIAM, Philadelphia, 1990.

9 S. Jaffard, Wavelet methods for fast resolution of elliptic problems, *SIAM J. Numer. Anal.,* 29: 965–986 (1992)

10 B. Jawerth and W. Sweldens, An overview of wavelet based multiresolution analysis, *Math. Rev.* to appear.

11 W. Lawton, Necessary and sufficient conditions for constructing orthonormal wavelet bases, *J. Math. Phys.,* 32: 57–61 (1991)

12 S. Mallat, Multifrequency channel decompositions of images and wavelet models, *IEEE Trans. Acoust. Speech Signal Process,* 37: 2091–2110 (1989)

13 S. Mallat, A theory for multiresolution signal decomposition: The wavelet representation, *IEEE Trans. on Patt. Anal. Mach. Intell.,* 11: 674–693 (1989)

14 Y. Maday, V. Perrier and J.-C. Ravel, Adaptativité dynamique sur bases d'ondelettes pour l'approximation d'équations aux dérivées partielles, *C. R. Acad. Sci. Paris, Série I,* 312: 405–410 (1991)

15 V. Perrier, Towards a method for solving partial differential equations using wavelet bases, in *Wavelets, Time-Frequency Methods and Phase Space,* Springer-Verlag, Berlin, 1989.

16 W.-C. Shann and C.-C. Yen, Quadratures involving polynomials and Daubechies' wavelets, Technical Report 9301, Department of Mathematics, National Central University, 1993.

17 G. Strang, Wavelets and dilation equations: A brief introduction, *SIAM Review,* 31: 614–627 (1989)

18 W. Sweldens and R. Piessens, Quadrature formulae for the calculation of the wavelet decomposition, *SIAM J. Numer. Anal.* to appear.

19 J. Xu and W.-C. Shann, Galerkin-wavelets methods for two-point boundary value problems, *Numer. Math.,* 63: 123–144 (1992)