# MULTIRESOLUTIONAL METHODS FOR DIFFERENCE EQUATIONS*

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**Abstract.** We develop a stable direct method with linear complexity for solving large banded sparse linear systems which are typical for the finite difference discretization of one-dimensional differential equations. This method was derived on the basic idea of multiresolutional decompositions that are fundamental to the theory of wavelets.

**Key words.** banded sparse matrix, differential equation, multiresolutional decomposition, wavelet

**Subject classifications:** AMS(MOS): 65F05, 65L12, 65L20, 65L60

**1. Introduction.** Fast algorithms for solving differential and integral equations are important for large-scale scientific computing. It usually brings down computational complexity for several orders of magnitude. This ability makes many difficult scientific and engineering problems become analyzable through computational methods.

Examples of fast algorithms are, for instance, multigrid methods, domain decomposition methods, multipole methods. Recently, a new direction of fast algorithms is being developed based on the wavelets theory.

Because of their similarities with Fourier transforms and finite elements, wavelets are studied as being a tool in the numerical solution of differential equations. Among the recent articles, we refer to the works by Bacry, Mallat and Papanicolaou [4], Beylkin, Coifman and Rokhlin [7], Glowinski, Lawton, Ravachol and Tenenbaum [16], Jaffard [17], Liandrat, Perrier and Tchamitchian [21] and Maday, Perrier and Ravel [22].

However, direct use of wavelets for representation of differential operators has the following disadvantages. First, the bandwidth of the associated stiffness matrices is wider as compared with that of the stiffness matrices associated with approximations of same order in finite difference or finite element. For instance, if Daubechies' wavelet of $p$ vanishing moments is used for the Galerkin approximation of the one dimensional Poisson's equation, the *a priori* $L^2$-norm error is of order $p$ [16]. The bandwidth can be illustrated by Table 1, where

$$D_k^p = \int \phi'(x)\phi'(x-k)\,dx.$$

Second, usual wavelets on real line cause problems as adapted for finite interval. Namely, they introduce fill-ins for a sparse matrix, and they bring a small eigenvalue that approaches to zero. See [27] for details. Although wavelets on finite intervals are derived recently (see Andersson, Hall, Jawerth and Peters [3], Cohen, Daubechies and Vial [9] and Mayer [25]), their numerical experiences and application to differential equations yet to be explored.

Third, the condition numbers of the stiffness matrices made by the scaling function of order $p$ (the values $D_k^p$ above) are large. An experiment for $p = 3$ showed that the condition numbers are $\approx 2500N^2$, where $N$ is the degree of freedom. This result is comparable to that in [7], where they

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| $D_k^p$ | $p=3$ | $p=4$ | $p=5$ |
|---|---|---|---|
| $k=0$ | 5.3571E−0 | 4.1659E−0 | 3.8349E−0 |
| $k=1$ | −3.3904E−0 | −2.6420E−0 | −2.4147E−0 |
| $k=2$ | 8.7619E−1 | 6.9786E−1 | 6.4950E−1 |
| $k=3$ | −1.1428E−1 | −1.5097E−1 | −1.8095E−1 |
| $k=4$ | −5.3571E−3 | 1.0572E−2 | 2.9907E−2 |
| $k=5$ | 0 | 1.6303E−3 | −7.9462E−4 |
| $k=6$ | 0 | −1.5921E−5 | −3.6714E−4 |
| $k=7$ | 0 | 0 | −1.6565E−6 |
| $k=8$ | 0 | 0 | −3.5387E−9 |

**Table 1.**

wrote the stiffness matrix under a periodic boundary condition in the standard form of wavelets basis. In that form, they found a diagonal preconditioner to control the condition numbers, such that a iterative method of solution can be effectively applied.

In this article, our approach is motivated by the multiresolution analysis in the wavelets theory. But we do not directly use wavelets. Our approach is based on a decomposition process such that the size of the problem can be reduced and this process can be recursive. More precisely, our method is a fast direct method for solving banded sparse linear systems, which are originated from discretizing differential equations by finite difference or finite element methods. The method is of linear computational complexity. Its application covers a wide range of differential equations.

There are others looking for alternatives that are orthogonal among the derivatives and are better adapted for bounded domains. Among them, we refer to those focus directly on the numerical solution of differential equations: Dahlke and Kunoth [11], Jawerth and Sweldens [19] and Xu and Shann [27].

**2. State of problem.** The second order finite element (similarly for finite difference) approximation of the differential equation

$$-u'' = f, \tag{2.1}$$

with a uniform mesh size $h$, is basically of the form

$$-u_{j-1} + 2u_j - u_{j+1} = f_j.$$

That for the equation

$$-(p(x)u')' + q(x)u = f, \quad p > 0, \quad q \geq 0, \tag{2.2}$$

is also a symmetric tridiagonal linear system with a matrix $A = (a_{ij})$ where

$$a_{j-1,j} = \int_{x_{j-1}}^{x_j} -p(x) + h^2 q(x)\,dx \quad a_{jj} = \int_{x_{j-1}}^{x_{j+1}} p(x)\,dx + h^2 q(x)\,dx.$$

Another set of examples come from hyperbolic conservation laws. For instance the backward Euler scheme for solving the linear advection equation

$$u_t + au_x = 0$$

is

$$u_j^{n+1} = u_j^n - \frac{a\,\Delta t}{2\,\Delta x}(u_{j+1}^{n+1} - u_{j-1}^{n+1}).$$

Thus we have $\frac{1}{2}A\boldsymbol{u}^{n+1} = \boldsymbol{u}^n$ with the matrix

$$A = \mathrm{diag}(-\sigma,\, 2,\, \sigma), \quad \sigma = \frac{a\,\Delta t}{\Delta x}.$$

In those examples mentioned above, we see that typically we have a linear system of equations with large and sparse matrix. The sparsity is naturally inherited from the locality of the differential operator. We shall assume that the involved matrices are basically banded matrices, for instance tridiagonal

$$A = \mathrm{diag}(a_j, b_j, c_j),$$

or penta-diagonal

$$A = \mathrm{diag}(a_j, b_j, c_j, d_j, e_j).$$

However, the bandedness may be affected by the associated boundary conditions. For instance, if we impose the periodic boundary condition to (2.1), together with the normalization $\int u\,dx = 0$. Then the matrix associated with a uniform discretization of (2.1) is

$$\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}_{N\times N}. \tag{2.3}$$

If the system with matrix (2.3) is to be solved by Gaussian elimination without pivoting, the last two equations will be

$$\begin{cases} \dfrac{N}{N-1}u_{N-1} - \dfrac{N}{N-1}u_N = \tilde{f}_{N-1} \\ \hfill N u_N = \tilde{f}_N. \end{cases} \tag{2.4}$$

Here

$$\tilde{f}_j = f_j + \frac{j-1}{j}\left(f_{j-1} + \frac{j-2}{j-1}\Big(f_{j-2} + \cdots + \frac{2}{3}(f_2 + \frac{1}{2}f_1)\cdots\Big)\right), \quad \text{for } j = 2, \ldots, N.$$

We shall solve $u_N$ by (2.4), then solve backward for $u_{N-1}, \ldots, u_1$. In our case $|u_j|$ are independent of $N$. We see that system (2.4) is very unbalanced, which is a typical sign of instability. For instance, if $N = 10^6$, then we will loss 6 significant digits in this calculation. This situation is usually resolved by the Gaussian elimination with pivoting. But this is not a desired approach because we will loss the sparsity and hence increase the computational complexity up to $O(N^3)$.

The goal in our article is to solve this kind of problem by a direct method with $O(N)$ computational complexity, and yet covers a wide range of applications like the foregoing examples.

**3. Basic idea.** The basic idea is the multiresolutional decomposition. Suppose the numerical solution $u_L$ of a differential equation is computed in a finite dimensional space $V_L$, we decompose

$$\begin{aligned} u_L &= u_{L-1} + (u_L - u_{L-1}) \\ &= u_{L-2} + (u_{L-1} - u_{L-2}) + (u_L - u_{L-1}) \\ &\;\;\vdots \\ &:= u_0 + w_0 + \cdots + w_{L-1}. \end{aligned}$$

Here $u_\ell$ is considered the resolution of $u_L$ at level $\ell$; while $w_\ell = u_{\ell+1} - u_\ell$ is the difference, or "detail," between the two resolutions. We want to design a sequence of spaces

$$V_0 \subset V_1 \subset \cdots \subset V_L$$

such that

$$V_\ell = V_0 + W_0 + \cdots + W_{\ell-1}$$

and $u_\ell \in V_\ell$, $w_\ell \in W_\ell$. The decomposition shall satisfy the following criteria.

Suppose $u_\ell$ is solved by the linear system $A_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell$; here $A_L = A$. After one step of decomposition, we shall have the linear system

$$\begin{pmatrix} A_{\ell-1} & B_{\ell-1} \\ C_{\ell-1} & D_{\ell-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\ell-1} \\ \boldsymbol{w}_{\ell-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{\ell-1} \\ \boldsymbol{g}_{\ell-1} \end{pmatrix}.$$

We would like the decomposition to be done in spaces $V_{\ell-1}$ and $W_{\ell-1}$ such that, ideally, (1) $B_{\ell-1} = 0$ or $C_{\ell-1} = 0$; (2) $D_{\ell-1}$ is diagonal or easy to be solved; and (3) $A_{\ell-1}$ keeps the same sparsity pattern as that of $A_\ell$, such that the process can be applied recursively.

In particular, for constant coefficient problems like (2.1), conditions (1) – (3) can be achieved and an example was classically known. Namely, the piecewise linear hierarchical finite element space. These particular basis functions are considers in [27] as a special case of a family of bases derived from anti-derivatives of Daubechies' wavelets.

For variable coefficient problems like (2.2), the criteria (1) – (3) can also be achieved. We can design the reduction such that $\boldsymbol{w}_{\ell-1}$ can be easily solved in terms of $\boldsymbol{u}_{\ell-1}$ and $\boldsymbol{g}_{\ell-1}$.

In the following sections, we will derive our reduction scheme by purely algebraic manipulations.

**4. The reduction scheme.** We shall start with the reduction scheme for matrices associated with the discretization of (2.2), which is tridiagonal. Then we will discuss the reduction scheme for general banded systems.

Suppose we are given a normalized tridiagonal system

$$a_j u_{j-1} + u_j + c_j u_{j+1} = f_j, \quad \text{for } j = 1, \ldots, N, \tag{4.1}$$

where $N$ is the number of unknowns which is assumed to be $2^L - 1$ for some positive integer $L$. The reduction process will reduce the system to a similar system with roughly half size in every reduction step. This process is then applied recursively until we arrive a very small system which can easily be solved. From the point of view of multiresolutional analysis, we define the number of unknowns at level $\ell$ to be $N_\ell = 2^\ell - 1$, $1 \le \ell \le L$. The approximate solution space $V_\ell$ at level $\ell$ is defined to be $\mathbb{R}^{N_\ell}$, whose elements is denoted by $u^\ell = (u_j^\ell)_{j=1}^{N_\ell}$. To illustrate our reduction process from level $\ell$ to level $\ell-1$, let us assume that at level $\ell$ we have a system of equations

$$a_j^\ell u_{j-1}^\ell + u_j^\ell + c_j^\ell u_{j+1}^\ell = f_j^\ell, \quad \text{for } j = 1, \ldots, N_\ell, \tag{4.2}$$

We can eliminate the odd-numbered variables and reduce the system to the even-numbered variables. Look at any three equations on level $\ell$:

$$\begin{array}{llllllll} a_{2j-1}^\ell u_{2j-2}^\ell + & u_{2j-1}^\ell & + c_{2j-1}^\ell u_{2j}^\ell & & & = f_{2j-1}^\ell & \cdots & (2j-1) \\ & a_{2j}^\ell u_{2j-1}^\ell + & u_{2j}^\ell & + c_{2j}^\ell u_{2j+1}^\ell & & = f_{2j}^\ell & \cdots & (2j) \\ & & a_{2j+1}^\ell u_{2j}^\ell + & u_{2j+1}^\ell & + c_{2j+1}^\ell u_{2j+2}^\ell & = f_{2j+1}^\ell & \cdots & (2j+1) \end{array}$$

If we combine them into

$$-a_{2j}^\ell \times (2j-1) + (2j) - c_{2j}^\ell \times (2j+1), \tag{4.3}$$

normalize (4.3) by $1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell$ and relabel the variables and coefficients, then we arrive the equation

$$a_j^{\ell-1} u_{j-1}^{\ell-1} + u_j^{\ell-1} + c_j^{\ell-1} u_{j+1}^{\ell-1} = f_j^{\ell-1}, \quad \text{for } j = 1, \ldots, N_{\ell-1}. \tag{4.4}$$

where $u_j^{\ell-1} \equiv u_{2j}^\ell$. The new set of coefficients can be derived from those in the higher level:

$$\begin{aligned} a_j^{\ell-1} &= -\frac{a_{2j}^\ell a_{2j-1}^\ell}{1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell} \\ c_j^{\ell-1} &= -\frac{c_{2j}^\ell c_{2j+1}^\ell}{1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell} \\ f_j^{\ell-1} &= \frac{f_{2j}^\ell - a_{2j}^\ell f_{2j-1}^\ell - c_{2j}^\ell f_{2j+1}^\ell}{1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell} \end{aligned} \tag{4.5}$$

Thus, at level $\ell$ if we define the change-of-variable

$$\begin{aligned} u_j^{\ell-1} &= u_{2j}^\ell \\ w_j^{\ell-1} &= a_{2j+1}^\ell u_{2j}^\ell + u_{2j+1}^\ell + c_{2j+1}^\ell u_{2j+2}^\ell \end{aligned} \tag{4.6}$$

in the domain of $A_\ell$ and the change-of-variable

$$\begin{aligned} f_j^{\ell-1} &= \frac{f_{2j}^\ell - a_{2j}^\ell f_{2j-1}^\ell - c_{2j}^\ell f_{2j+1}^\ell}{1 - a_{2j}^\ell c_{2j-1}^\ell - a_{2j+1}^\ell c_{2j}^\ell} \\ g_j^{\ell-1} &= f_{2j+1}^\ell \end{aligned} \tag{4.7}$$

in the range of $A_\ell$, then the system $A_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell$ is reduced to

$$\begin{pmatrix} A_{\ell-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\ell-1} \\ \boldsymbol{w}_{\ell-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{\ell-1} \\ \boldsymbol{g}_{\ell-1} \end{pmatrix}.$$

Thus, the change-of-variable:

$$\begin{aligned} u^L &\longleftrightarrow (u^1, w^1, w^2, \cdots, w^{L-1}) \\ f^L &\longleftrightarrow (f^1, g^1, g^2, \cdots, g^{L-1}) \end{aligned}$$

diagonalizes the system. From $(u^1, w^1, w^2, \cdots, w^{L-1})$ we can find $u^L$ recursively by

$$\begin{aligned} u^\ell_{2j} &= u^{\ell-1}_j \\ u^\ell_{2j+1} &= w^{\ell-1}_j - a^\ell_{2j+1} u^{\ell-1}_j - c^\ell_{2j+1} u^{\ell-1}_{j+1}. \end{aligned}$$

For the direct methods, we are concerned with the stability.

In case of variable coefficients, we define a mapping $\mathcal{F}(\boldsymbol{a}, \boldsymbol{c}) \to (\hat{\boldsymbol{a}}, \hat{\boldsymbol{c}})$. Here $\boldsymbol{a}$, etc, are the vectors of $(a_j)$, etc, respectively.

THEOREM 4.1. *An invariant region of the mapping $\mathcal{F}$ whose action is defined as in* (4.5) *is*

$$D := \Big\{ (\boldsymbol{a}, \boldsymbol{c}) \mid |a_j| + |c_j| \le 1 \Big\}.$$

*Moreover, if $(\boldsymbol{a}, \boldsymbol{c})$ is an interior point of $D$, then $\mathcal{F}^k(\boldsymbol{a}, \boldsymbol{c})$ converges to $(\mathbf{o}, \mathbf{o})$ as $k \to \infty$.*

*Proof.* From $|c^\ell_{2j-1}|$, $|a^\ell_{2j+1}| \le 1$ and $|a^\ell_{2j}| + |c^\ell_{2j}| \le 1$, we yield

$$1 - a^\ell_{2j} c^\ell_{2j-1} - a^\ell_{2j+1} c^\ell_{2j} \ge 0.$$

Using this and

$$|a^\ell_k| + |c^\ell_k| \le 1, \ k = 2j-1, \ 2j, \ 2j+1,$$

we have

$$\begin{aligned} & |a^\ell_{2j}|(|a^\ell_{2j-1}| + |c^\ell_{2j-1}|) + |c^\ell_{2j}|(|a^\ell_{2j+1}| + |c^\ell_{2j+1}|) \le 1 \\ \Longleftrightarrow\ & |a^\ell_{2j} a^\ell_{2j-1}| + |c^\ell_{2j} c^\ell_{2j+1}| \le 1 - |a^\ell_{2j} c^\ell_{2j-1}| - |a^\ell_{2j+1} c^\ell_{2j}| \\ \Longrightarrow\ & |a^\ell_{2j} a^\ell_{2j-1}| + |c^\ell_{2j} c^\ell_{2j+1}| \le 1 - a^\ell_{2j} c^\ell_{2j-1} - a^\ell_{2j+1} c^\ell_{2j} \\ \Longleftrightarrow\ & \frac{|a^\ell_{2j} a^\ell_{2j-1}| + |c^\ell_{2j} c^\ell_{2j+1}|}{1 - a^\ell_{2j} c^\ell_{2j-1} - a^\ell_{2j+1} c^\ell_{2j}} \le 1 \\ \Longleftrightarrow\ & |a^{\ell-1}_j| + |c^{\ell-1}_j| \le 1. \end{aligned}$$

Suppose $\max_j(|a^\ell_j| + |c^\ell_j|) = \rho_\ell < 1$ and suppose

$$\max_j |a^{\ell-1}_j| + |c^{\ell-1}_j| = |a^{\ell-1}_{j_0}| + |c^{\ell-1}_{j_0}| = \rho_{\ell-1}.$$

Then, from the above argument $\rho_{\ell-1} \le 1$. Further,

$$\begin{aligned} & \rho_{\ell-1}(1 - a^\ell_{2j_0} c^\ell_{2j_0-1} - a^\ell_{2j_0+1} c^\ell_{2j_0}) = |a^\ell_{2j_0} a^\ell_{2j_0-1}| + |c^\ell_{2j_0} c^\ell_{2j_0+1}| \\ \Longrightarrow\ & \rho_{\ell-1} \le |a^\ell_{2j_0}|(|a^\ell_{2j_0-1}| + |c^\ell_{2j_0-1}|) + |c^\ell_{2j_0}|(|a^\ell_{2j_0+1}| + |c^\ell_{2j_0+1}|) \le \rho^2_\ell \end{aligned}$$

Hence, $\mathcal{F}^k(\boldsymbol{a}, \boldsymbol{c})$ converges to zero quadratically as $k \to \infty$, provided $\rho_L < 1$ initially. □

In case of constant coefficients, the reduction amounts to the mapping

$$F : (a, c) \longmapsto (\hat{a}, \hat{c}) = \left(-\frac{a^2}{1 - 2ac}, \ -\frac{c^2}{1 - 2ac}\right). \tag{4.8}$$

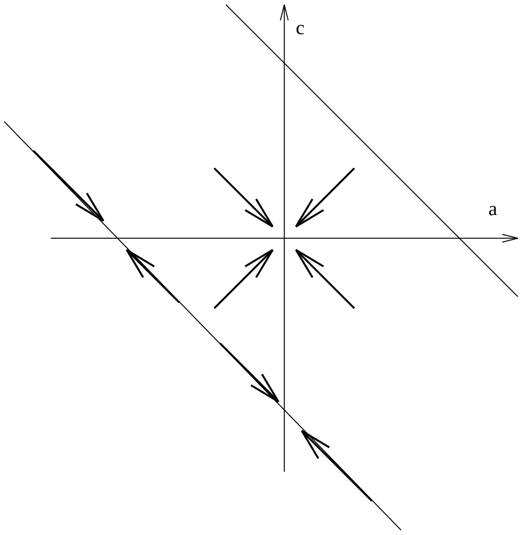


**Figure 1.** *Diagram for $F(a, c)$.*

The invariant and asymptotic behaviors of $F$ are summarized in the following theorem.

THEOREM 4.2. *The mapping $F$ in (4.8) has the invariant region*

$$D := |a + c| \leq 1.$$

*If $(a, c)$ is in the interior of $D$, then $F^k(a, c) \to (0, 0)$ quadratically as $k \to \infty$. $(-\frac{1}{2}, -\frac{1}{2})$ is an unstable fixed point; $(-1, 0)$ and $(0, -1)$ are saddle points. They are illustrated in Figure 1.*

*Proof.* We notice that $|a + c| \leq 1$ implies $1 - 2ac > 0$. Hence,

$$\begin{aligned} & |\hat{a} + \hat{c}| \leq 1 \\ \iff & a^2 + c^2 \leq 1 - 2ac \\ \iff & |a + c| \leq 1. \end{aligned}$$

The properties of fixed points follow from linear analysis for $F$. □

For matrix which is not diagonal dominant, we have the following theorem. Its proof is straight forward.

THEOREM 4.3. *The region*

$$D := \Big\{(\boldsymbol{a}, \boldsymbol{c}) \mid a_j a_{j+1} \geq 0,\ c_j c_{j+1} \geq 0,\ a_j c_{j-1} \leq \frac{1}{4},\ |a_j + c_j| \leq 1\Big\}.$$

*is invariant under the mapping $\mathcal{F}$ whose action is defined as in* (4.5)

When $(a_j^\ell, 1, c_j^\ell)$ are in the invariant region $D$, the reduction scheme remains stable. The matrices associated with all model problems proposed in Section 2 are within the invariant regions. Therefore the reduction scheme (4.5) is a stable direct method for the solution of those linear systems.

We can further generalize this reduction scheme for wider banded matrices. To illustrate the procedure, we let $A_\ell = \mathrm{diag}(a, b, 1, c, d)$ be a penta-diagonal matrix with constant entries. The system of equations is like

$$
\begin{pmatrix}
 & \ddots & & & & & & & \\
a & b & 1 & c & d & & & & \\
 & a & b & 1 & c & d & & & \\
 & & a & b & 1 & c & d & & \\
 & & & a & b & 1 & c & d & \\
 & & & & a & b & 1 & c & d \\
 & & & & & & & \ddots &
\end{pmatrix}
\begin{pmatrix}
\vdots \\ u^{\ell}_{2j-2} \\ u^{\ell}_{2j-1} \\ u^{\ell}_{2j} \\ u^{\ell}_{2j+1} \\ u^{\ell}_{2j+2} \\ \vdots
\end{pmatrix}
= \boldsymbol{f}_{\ell}. \tag{4.9}
$$

Suppose the following $4 \times 4$ linear system has a unique solution:

$$
\begin{pmatrix}
b & a & & \\
c & 1 & a & \\
 & d & 1 & b \\
 & & d & c
\end{pmatrix}
\begin{pmatrix}
\alpha \\ \beta \\ \gamma \\ \delta
\end{pmatrix}
=
\begin{pmatrix}
0 \\ -b \\ -c \\ 0
\end{pmatrix}.
$$

Multiply (4.9) by the matrix of the following form

$$
\begin{pmatrix}
 & \ddots & & & & & & & \\
\alpha & \beta & 1 & \gamma & \delta & & & & \\
 & 1 & & & & & & & \\
 & & \alpha & \beta & 1 & \gamma & \delta & & \\
 & & & 1 & & & & & \\
 & & & & \alpha & \beta & 1 & \gamma & \delta \\
 & & & & & & & \ddots &
\end{pmatrix}.
$$

Then the system (4.9) will be like

$$
\begin{pmatrix}
 & \ddots & & & & & & & \\
\hat{b} & & 1 & & \hat{c} & & \hat{d} & & \\
 & a & b & 1 & c & d & & & \\
\hat{a} & & \hat{b} & & 1 & & \hat{c} & & \hat{d} \\
 & & & a & b & 1 & c & d & \\
 & & \hat{a} & & \hat{b} & & 1 & & \hat{c} \\
 & & & & & & & \ddots &
\end{pmatrix}
\begin{pmatrix}
\vdots \\ u^{\ell}_{2j-2} \\ u^{\ell}_{2j-1} \\ u^{\ell}_{2j} \\ u^{\ell}_{2j+1} \\ u^{\ell}_{2j+2} \\ \vdots
\end{pmatrix}
= \hat{\boldsymbol{f}}_{\ell}. \tag{4.10}
$$

Permute the above system by odd/even rows and columns, let

$$
\begin{aligned}
u^{\ell-1}_{j} &= u^{\ell}_{2j}, \\
w^{\ell-1}_{j} &= a u^{\ell}_{2j-3} + u^{\ell}_{2j-1} + d u^{\ell}_{2j+1}.
\end{aligned}
$$

Then we can reduce (4.10) into

$$
\begin{pmatrix}
A_{\ell-1} & 0 \\
C_{\ell-1} & I
\end{pmatrix}
\begin{pmatrix}
\boldsymbol{u}_{\ell-1} \\ \boldsymbol{w}_{\ell-1}
\end{pmatrix}
=
\begin{pmatrix}
\boldsymbol{f}_{\ell-1} \\ \boldsymbol{g}_{\ell-1}
\end{pmatrix},
$$

where $A_{\ell-1} = \mathrm{diag}(\hat{a}, \hat{b}, 1, \hat{c}, \hat{d})$ and $C_{\ell-1}$ has two bands with $b$ and $c$. Once $\boldsymbol{u}_{\ell-1}$ is solved, the even-numbered terms of $\boldsymbol{u}_\ell$ are readily known. The rest can be obtained by solving the tridiagonal system

$$\mathrm{diag}(a, 1, d)\,(\text{odd-numbered terms of } \boldsymbol{u}_\ell) = C_{\ell-1}\, \boldsymbol{g}_{\ell-1}.$$

**5. Relations to hierarchical bases and wavelets.** Now we suppose the differential equation is defined in the spatial domain $(0,1)$, on which we define a sequence of partitions. For simplicity of notations (which is not theoretically essential), we let $h_\ell = 2^{-\ell}$, for $\ell = 1, \ldots, L$; here $L > 0$ is the prescribed highest resolution. The degree of freedom for each level $\ell$ is $N_\ell$. Define the partitions $\mathcal{T}_\ell = \{x_k^\ell\}$ such that $x_k^\ell = k\, h_\ell$, where $k = 1, \ldots, N_\ell$. Let $\phi_k^\ell(x)$ be the piecewise linear nodal basis function associated with $x_k^\ell$ in $\mathcal{T}_\ell$. That is, $\phi_k^\ell(x_j^\ell) = \delta_{jk}$ and $\phi_k^\ell(x)$ is a linear polynomial within each $[x_{j-1}^\ell, x_j^\ell]$. We denote the nodal basis for $V_L$ by $\mathcal{N}$.

The stiffness matrix for problem (2.1) derived from nodal basis $\mathcal{N}$ can be normalized to $\mathrm{diag}(a, 1, c)$, where $(a, c) = (-\frac{1}{2}, -\frac{1}{2})$ is a fixed point for the reduction mapping (4.8).

Now we define the hierarchical basis associated with $\mathcal{N}$:

$$\psi_k^\ell(x) = \sqrt{\frac{h_\ell}{2}}\phi_{2k-1}^{\ell+1}(x), \quad \text{for } \ell = 0, \ldots, L-1 \text{ and } k = 1, \ldots, N_\ell + 1.$$

We denote the hierarchical basis by $\mathcal{H}$. One can verify that $|\mathcal{N}| = |\mathcal{H}|$ and $V_L = \mathrm{span}\mathcal{H}$. That is, $\mathcal{H}$ consists of another basis for $V_L$.

Here, under the multiresolutional point of view, we do not change the basis from $\mathcal{N}$ to $\mathcal{H}$ all at once. Let $V_\ell$ be the subspace of $V_L$ spanned by the nodal basis functions on $\mathcal{T}_\ell$, $W_\ell$ be that spanned by the hierarchical basis $\psi_k^\ell$ at the level $\ell$. Then $V_\ell = V_{\ell-1} + W_{\ell-1}$. That is, for any $u_\ell \in V_\ell$, we have

$$u_\ell = \sum u_k^\ell \phi_k^\ell(x) = \sum u_k^{\ell-1}\phi_k^{\ell-1}(x) + \sum w_k^{\ell-1}\psi_k^{\ell-1}(x). \tag{5.1}$$

In this case our reduction scheme $\boldsymbol{u}_\ell \to (\boldsymbol{u}_{\ell-1}, \boldsymbol{w}_{\ell-1})$ corresponds to the decomposition from $V_\ell$ into $V_{\ell-1} + W_{\ell-1}$. As we have remarked in Section 3.

In general, the spaces $V_\ell$ and $W_\ell$ are conceptually defined and we can postulate a set of basis for each of them. Then our change of variables implies a change of basis as in (5.1). From here we can derive the scaling equations

$$\phi_0^{\ell-1}(x) = \sum_k c_k^\ell \phi_k^\ell(x).$$

However, the scaling functions $\phi_\cdot^\ell(x)$ applied on each level of the multiresolution are, in general, different.

To simplify the notation, let us concencrate on the relation between two levels. For general $\mathrm{diag}(a, 1, c)$, we consider the cases that induce the scaling equation

$$\phi(x) = \alpha\phi(2x) + \phi(2x-1) + (1-\alpha)\phi(2x-2). \tag{5.2}$$

We want a solution $\phi$ with support in $[0, 2]$. It is shown by Lau and Wang [20] that (5.2) has non-zero $L^p$-solution if and only if

$$\tfrac{1}{2}(|\alpha|^p + |1-\alpha|^p) < 1.$$

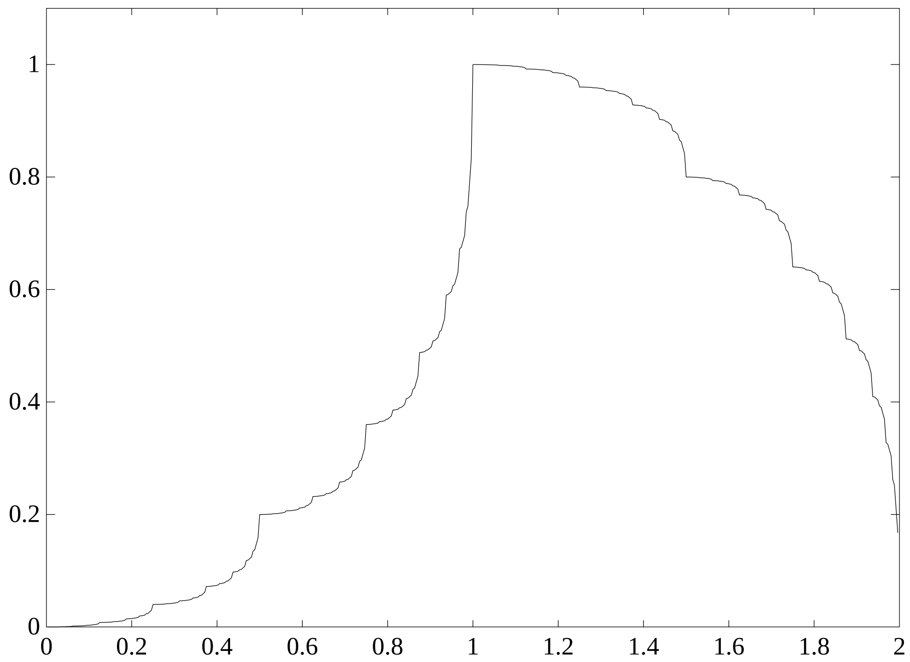


**Figure 2.** $\phi(x)$ as in (5.2) and $\alpha = 0.2$.

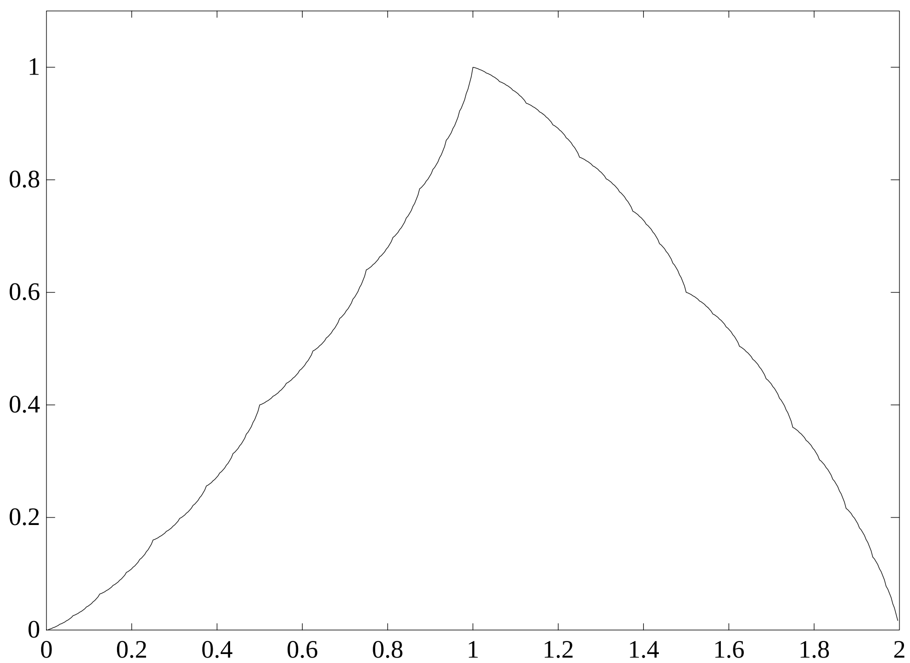


**Figure 3.** $\phi(x)$ as in (5.2) and $\alpha = 0.4$.

It is interesting to observe the pictures of $\phi(x)$ that satisfies (5.2). We note that when $\alpha = 1$, it reduces to the Haar function on $[0,1]$; while $\alpha = 1/2$ corresponds to the spline $1 - |x-1|$ on $[0,2]$; and $\alpha = 0$ reduces to the translation of Haar function on $[1,2]$. See Figures 2 and 3 for some other situations.

It is also interesting to compute the $2\pi$-periodic characteristic functions of the scaling functions. Namely the function $m_0(\xi)$ that $\hat{\phi}(\xi) = m_0(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2})$.

Let us take the example of the reduction of constant penta-diagonal matrix. With the special case of $\mathrm{diag}(a, b, 1, b, a)$, the change of variables in the reduction scheme induces the following change of basis

$$\begin{aligned} \phi^{\ell}_{2j} &= \phi^{\ell-1}_{j} + b\psi^{\ell-1}_{j} + b\psi^{\ell-1}_{j-1}, \\ \phi^{\ell}_{2j+1} &= a\psi^{\ell-1}_{j+1} + a\psi^{\ell-1}_{j-1} + \psi^{\ell-1}_{j}. \end{aligned}$$

To simplify our notation, we focus on one level and write

$$\begin{aligned} \phi(2x) &= \phi(x) + b\psi(x) + b\psi(x-1), \\ \psi(2x+1) &= a\psi(x+1) + a\psi(x-1) + \psi(x). \end{aligned}$$

Take the Fourier transform on both sides of the above equations, we can solve $\hat{\phi}(\xi)$ for

$$\hat{\phi}(\xi) = \frac{1}{2}\left[1 - \frac{2b\cos\frac{\xi}{2}}{1 + 2a\cos\xi}\right]\hat{\phi}(\frac{\xi}{2}).$$

If $a + b = -\frac{1}{2}$, we find $m_0(\pi) = m_0'(\pi) = 0$. That is, polynomials $\{1, x\}$ can be spanned by $\phi(x-k)$. When $a = 1/6$, $b = -2/3$, $\mathrm{diag}(a, b, 1, b, a)$ is the stiffness matrix associated with the biharmonic operator $u^{(4)}$, and this matrix is invariant under our reduction process. The corresponding scaling function is not compactly supported, it vanishes at nonzero integers. The essential portion of this scaling function is shown in Figure 4.

**6. Conclusion.** As we argued in the first section, the "usual" wavelets such as those in [12] do not show much advantage for solving differential equations. What essential are the concepts of orthogonal on the derivative level, the multiresolutional decomposition and the compact supportedness (locality) which can also fit the boundary conditions.

Here we demonstrate a result in this way of thinking. The idea of multiresolution is used to reduce the linear system to half size. If the band-width of the matrix is independent of $N$, usually this means the problem is one-dimensional, this method gives a direct method of complexity $O(N)$ and it is stable for many applicable physical problems.

REFERENCES

[1] B. Alpert, *Construction of simple multiscale bases for fast matrix operations,* in "Wavelets and their applications," Jones and Barlett Publishers, Boston, 1992, pp. 211–226.

[2] B. Alpert, G. Beylkin, R. Coifman and V. Rokhlin, *Wavelet-like bases for the fast solution of second-kind integral equations,* SIAM J. Sci. Comp., 14 (1993), pp. 159–184.

[3] L. Andersson, N. Hall, B. Jawerth and G. Peters, *Wavelets on closed subsets of the real line,* in "Recent advances in wavelet analysis," Academic Press, Boston, 1994, pp. 1–61.

[4] E. Bacry, S. Mallat and G. Papanicolaou, *A wavelet based space-time adaptive numerical method for partial differential equations,* RAIRO Math Modelling Numer. Anal., 26 (1992), pp. 793–834.

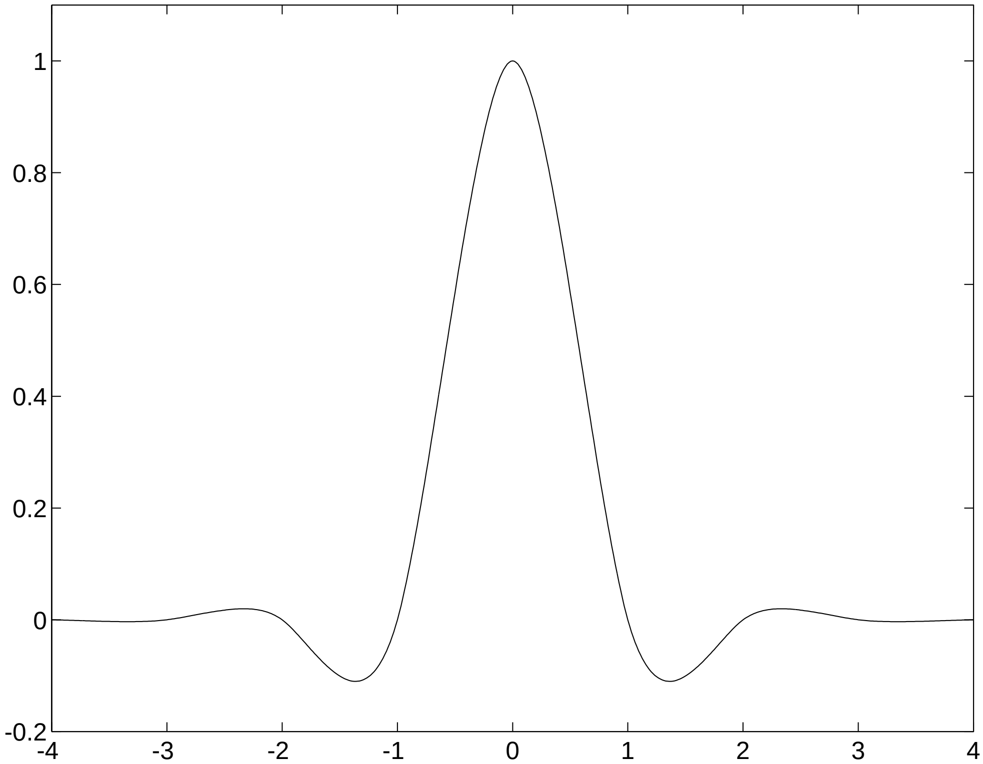


**Figure 4.** $\phi(x)$ corresponds to $\mathrm{diag}(\frac{1}{6}, -\frac{2}{3}, 1, -\frac{2}{3}, \frac{1}{6})$.


[5] G. Battle, *Wavelets: A renormalization group point of view,* in "Wavelets and their applications," Jones and Barlett Publishers, Boston, 1992, pp. 323–349.

[6] G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms,* I, Comm. Pure Appl. Math., 44 (1991), pp. 141–183.

[7] ____, *Wavelets in numerical analysis,* in "Wavelets and their applications," Jones and Barlett Publishers, Boston, 1992, pp. 181–210.

[8] I-L. Chern, M.-H. Chou and W.-C. Shann, *Wavelets and multiscale computational methods,* Project Report NSC 83-0208-M-008-052, November, 1993.

[9] A. Cohen, I. Daubechies and P. Vial, *Wavelets and fast wavelet transforms on the interval,* Appl. Comput. Harmonic Anal. to a ppear.

[10] R. Coifman, Y. Meyer, S. Quake and V. Wickerhauser, *Signal processing and compression with wavelet packets,* preprint, Numerical Algorithms Research Group, Yale University, 1990.

[11] S. Dahlke and A. Kunoth, *A biorthogonal wavelet approach for solving boundary value problems,* Technical report, Institut für Geometrie und Praktische Mathematik, 1993.

[12] I. Daubechies, *Orthonormal bases of compactly supported wavelets,* Comm. Pure Appl. Math., 41 (1988), pp. 909–996.

[13] ____, *The wavelet transform, time-frequency localization and signal analysis,* IEEE Trans. Info. Theory, 36 (1990), pp. 961–1005.

[14] ____, "Ten lectures on wavelets," SIAM, Philadelphia, 1992.

[15] D. L. Donoho and I. M. Johnstone, *Minimax estimation via wavelet shrinkage,* Technical Report, Department of Statistics, Stanford University, 1992.

[16] R. Glowinski, W. M. Lawton, M. Ravachol and E. Tenenbaum, *Wavelets solution of linear and nonlinear elliptic, parabolic and hyperbolic problems in one space dimension,* in

"Computing methods in applied sciences and engineering," SIAM, Philadelphia, 1990, pp. 55–120.

[17] S. Jaffard, *Wavelet methods for fast resolution of elliptic problems,* SIAM J. Numer. Anal., 29 (1992), pp. 965–986.

[18] B. Jawerth and W. Sweldens, *An overview of wavelet based multiresolution analysis,* Math. Rev. to appear.

[19] ———, *Wavelet multiresolution analyses adapted for the fast solution of boundary value ordinary differential equations,* in "Proceedings of the 6th Copper Mountain Multigrid Conference," NASA Conference Pub, 1993, pp. 259–273.

[20] K-S Lau and J. Wang, *Characterization of $L^p$-solutions for the two-scale dilation equations,* SIAM J. Math. Anal. to appear.

[21] J. Liandrat, V. Perrier and Ph. Tchamitchian, *Numerical resolution of nonlinear partial differential equations using the wavelet approach,* in "Wavelets and their applications," Jones and Barlett Publishers, Boston, 1992, pp. 211–226.

[22] Y. Maday, V. Perrier and J.-C. Ravel, *Adaptativité dynamique sur bases d'ondelettes pour l'approximation d'équations aux dérivées partielles,* C. R. Acad. Sci. Paris, Série I, 312 (1991), pp. 405–410.

[23] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$,* Trans. Amer. Math. Soc., 315 (1989), pp. 69–87.

[24] ———, *A theory for multiresolution signal decomposition: The wavelet representation,* IEEE Trans. on Patt. Anal. Mach. Intell., 11 (1989), pp. 674–693.

[25] Y. Meyer, *Ondelettes sur l'intervalle,* Rev. Math. Iberoamer., 7 (1992), pp. 115–133.

[26] G. Strang, *Wavelets and dilation equations: A brief introduction,* SIAM Review, 31 (1989), pp. 614–627.

[27] J. Xu and W. Shann, *Galerkin-wavelets methods for two-point boundary value problems,* Numer. Math., 63 (1992), pp. 123–144.