Exact Solutions for Daubechies Orthonormal Scaling Coefficients*

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Abstract. We review the procedure for the computation of the filter coefficients for the Daubechies' orthonormal scaling functions. It is well known that there are exact solutions for these coefficients of length 2, 4, and 6. Now we can construct the exact solutions for those of length 8 and 10.

Let $\{c_0, c_1, \ldots, c_{2p-1}\}$ be the scaling coefficients for Daubechies' orthonormal scaling functions of order p ($p \geq 1$). Define the frequency response function

$$m_0(\xi) = \frac{1}{2} \sum_{k=0}^{2p-1} c_k e^{-ik\xi}.$$

In the *Ten Lectures* [1], it is derived that a necessary condition for orthonormality is

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1. \tag{1}$$

The approximation property amounts to specify that $m_0(\xi)$ is of the form

$$m_0(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^p \mathcal{L}(\xi), \tag{2}$$

where $\mathcal{L}$ is a trigonometric polynomial. Condition (1) alone is not sufficient for the construction of orthonormal scaling function $\phi(x)$, which is defined by

$$\hat{\phi}(\xi) = \prod_{\ell=1}^{\infty} m_0(\frac{\xi}{2^\ell}).$$

For a detailed account of the sufficient conditions, see Section 6.3 of [1] or the technical report by Shann and Yen [2].

In order to construct $m_0(\xi)$ of the form (2) and satisfies (1), we have to know $\mathcal{L}(\xi)$. In [1], $\mathcal{L}(\xi)$ is constructed by the procedure of *extracting the square root* (or, in the engineering literature, *spectral factorization*) from $|\mathcal{L}(\xi)|^2$. It is shown that

$$|\mathcal{L}(\xi)|^2 = P(\sin^2 \frac{\xi}{2}), \tag{3}$$

with

$$P(y) = P_p(y) + y^p R(\frac{1}{2} - y),$$

where

$$P_p(y) = \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k$$

and $R(\cdot)$ is an odd polynomial chosen such that $P(y) \geq 0$ for all $y \in [0, 1]$.

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The choice of Daubechies is for the "shortest filter length," which amounts to taking $R = 0$. With this choice, $P(y)$ is a polynomial of degree $p-1$, and $P(\sin^2\frac{\xi}{2})$ can be written in the form

$$\sum_{k=0}^{2p-2} a_k \cos k\xi$$

with real coefficients a_k. And that can be further transformed to a polynomial of degree $2p-2$ in terms of $\cos\xi$. To "extract the square root" by means of Lemma 6.1.3 in [1], one has to factorize this polynomial of degree $2p-2$, and choose among the roots to construct a "square root" $\mathcal{L}(\xi)$. In general the "square roots" are non-unique. If we ignore the reflections of the solutions (that is, consider $\{c_{2p-2},\cdots,c_0\}$ the same set of coefficients as $\{c_0,\cdots,c_{2p-2}\}$), then there are $2^{\lfloor\frac{p}{2}\rfloor-1}$ different choices of $\mathcal{L}(\xi)$ out of $|\mathcal{L}(\xi)|^2$. And thus there are $2^{\lfloor\frac{p}{2}\rfloor-1}$ different choices of orthonormal scaling functions of order p. The "normal" choice is to retain the zeros symmetric within the unit circle. This choice corresponds to the "minimum phase filter," and the corresponding scaling functions are very asymmetric. If there are more than one choice (for $p \ge 4$), it is possible to choose among them the "least asymmetric" scaling function. See Section 8.1.1, Table 6.3 and Figure 6.4 in [1].

There are unique sets of the scaling coefficients in the cases of $p = 1$, 2 and 3. And we know the exact values of these coefficients, namely, for $p = 1$,

$$1, \quad 1;$$

for $p = 2$,

$$\frac{1+\sqrt{3}}{4}, \quad \frac{3+\sqrt{3}}{4}, \quad \frac{3-\sqrt{3}}{4}, \quad \frac{1-\sqrt{3}}{4};$$

and for $p = 3$,

$$\frac{1+\sqrt{10}+\theta}{16}, \frac{5+\sqrt{10}+3\theta}{16}, \frac{5-\sqrt{10}+\theta}{8}, \frac{5-\sqrt{10}-\theta}{8}, \frac{5+\sqrt{10}-3\theta}{16}, \frac{1+\sqrt{10}-\theta}{16},$$

where $\theta = \sqrt{5+2\sqrt{10}}$. (Here we normalize the coefficients such that $\sum c_k = 2$.)

For the cases of $p \ge 4$, the approach suggested in [1] would have difficulties for the exact solutions. Because that will involve the factorization of a polynomial of degree $2p-2 \ge 6$. (The Galois theory implies that there is no procedure to solve a polynomial of degree ≥ 5.) Therefore we turn to the numerical solutions in general. Although the exact solutions of these scaling coefficients have little value in terms of application, it is still an interesting exercise and a theoretical curiosity to work out the details.

In the following, we will present ways to obtain the exact values of the scaling coefficients of order $p = 4$ and $p = 5$. Before that, let's define a few new notation. Let $z = e^{-i\xi}$. Take the z-transform of the frequency response function $m_0(\xi)$, we have

$$H(z) = \frac{1}{2}\sum_{k=0}^{2p-1} c_k z^k = C(1+z)^p L(z), \tag{4}$$

where

$$L(z) = \sum_{k=0}^{p-1} a_k z^k$$

is a polynomial of degree $p-1$ and C is determined such that $H(1) = 1$. Define

$$Q(z) = \sum_{k=0}^{p-1} \binom{p-1+k}{k} \left(\frac{2-z-z^{-1}}{4}\right)^k.$$

Note that $z^{p-1}Q(z)$ is a polynomial of degree $2p-2$ with symmetric coefficients, and it is the z-transform of the polynomial $P(\sin^2 \frac{\xi}{2})$ in (3). We will have to determine $L(z)$ such that $z^{p-1}Q(z) = L(z)\overline{L(z)}$.

Since $|z| = 1$ and we restrict ourselves to the real valued a_k,

$$\overline{L(z)} = \sum_{k=0}^{p-1} a_k \bar{z}^k = \sum_{k=0}^{p-1} a_k z^{-k} = \frac{1}{z^{p-1}} \sum_{k=0}^{p-1} a_k z^{p-1-k}.$$

We can determine a_k by comparing the coefficients of

$$z^{p-1}Q(z) = (a_0 + a_1 z + \cdots + a_{p-1}z^{p-1})(a_0 z^{p-1} + \cdots + a_{p-2}z + a_{p-1}).$$

The case of $\mathbf{p = 4}$

In this case

$$z^3 Q(z) = -5 + 40z - 131z^2 + 208z^3 - 131z^4 + 40z^5 - 5z^6$$

and $C = \frac{1}{64}$ in (4). We would like to solve the four unknowns a_0, ..., a_3. Since $Q(z)$ is symmetric, it amounts to the solution of a system of four quadratic equations:

$$\begin{aligned} a_0 a_3 &= -5 \\ a_0 a_2 + a_1 a_3 &= 40 \\ a_0 a_1 + a_1 a_2 + a_2 a_3 &= -131 \\ a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 208 \end{aligned}$$

We can use these four equations to obtain the values of $(a_0+a_1+a_2+a_3)^2$ and $(a_0-a_1+a_2-a_3)^2$. The trick is that we can replace two of the foregoing equations by $a_0 + a_1 + a_2 + a_3 = 4$ and $a_0 - a_1 + a_2 - a_3 = 2\sqrt{140}$. Then, we can reduce the above system of equations to a parameter α:

$$a_0 = \alpha \quad a_1 = 2 - \sqrt{140} + \frac{5}{\alpha} \quad a_2 = 2 + \sqrt{140} - \alpha \quad a_3 = -\frac{5}{\alpha}$$

where α is a root of the following polynomial:

$$\alpha^4 - 2(1+\sqrt{35})\alpha^3 + 40\alpha^2 + 10(1-\sqrt{35})\alpha + 25 = 0.$$

There is an algebraic formula for the solution of polynomials of degree 4, and a computer algebra system, like `Mathematica`, knows how to do it.

There are a pair of conjugate complex roots and two different real roots. The real root

$$\alpha_1 = \frac{1}{2} + \frac{1}{2}\sqrt{35} + \frac{1}{6}\sqrt{\frac{3v}{u^{1/3}}} + \frac{1}{6}\sqrt{3s},$$

which is approximately 10.4257, gives the "minimum phase" orthonormal scaling function in [1] (this is the "normal" choice). The other real root

$$\alpha_2 = \frac{1}{2} + \frac{1}{2}\sqrt{35} + \frac{1}{6}\sqrt{\frac{3v}{u^{1/3}}} - \frac{1}{6}\sqrt{3s},$$

which is approximately 1.4583, gives the "least asymmetric" orthonormal scaling function in [1]. For both of the above expressions,

$$\begin{aligned} u &= 700 + 210\sqrt{15}, \\ v &= 28\,u^{1/3} - 70 \cdot 2^{1/3} + (2u)^{2/3} + 6\,u^{1/3}\sqrt{35}, \\ s &= \frac{56\,u^{1/3}\sqrt{v} + 70 \cdot 2^{1/3}\sqrt{v} - (2u)^{2/3}\sqrt{v} + 12u^{1/3}\sqrt{35v} + 336\sqrt{3u} + 48\sqrt{105u}}{u^{1/3}\sqrt{v}}. \end{aligned}$$

Once we have (two sets of) the values of a_k, the (two sets of) scaling coefficients c_k are determined by

$$(c_0, c_1, \ldots, c_7) = \frac{1}{32}(1,4,6,4,1) * (a_0, a_1, a_2, a_3),$$

where $*$ means the discrete convolution.

The case of p = 5

In this case

$$z^4Q(z) = 35 - 350z^1 + 1520z^2 - 3650z^3 + 5018z^4 - 3650z^5 + 1520z^6 - 350z^7 + 35z^8$$

and $C = \frac{1}{256\sqrt{2}}$ in (4). The trick is to write

$$\begin{aligned} Q(z) &= 35(z+\frac{1}{z})^4 - 350(z+\frac{1}{z})^3 + 1380(z+\frac{1}{z})^2 - 2600(z+\frac{1}{z}) + 2048 \\ &= 35(z+\frac{1}{z}-\alpha)(z+\frac{1}{z}-\bar{\alpha})(z+\frac{1}{z}-\beta)(z+\frac{1}{z}-\bar{\beta}). \end{aligned}$$

Again, a computer algebra system fits in here and solves α and β:

$$\begin{aligned} \alpha = {} & \frac{5}{2} + \frac{1}{70}\sqrt{35v} \\ & - \frac{1}{70}\sqrt{35}\left[(-90\,u^{1/3}\sqrt{v} - 480\cdot 2^{1/3}\sqrt{v} - 4\,(2u)^{2/3}\sqrt{v} - 150\sqrt{35}\,u^{1/3})/(u^{1/3}\sqrt{v})\right]^{1/2} \end{aligned}$$

and

$$\begin{aligned} \beta = {} & \frac{5}{2} + \frac{1}{70}\sqrt{35v} \\ & - \frac{1}{70}\sqrt{35}\left[(-90\,u^{1/3}\sqrt{v} - 480\cdot 2^{1/3}\sqrt{v} - 4\,(2u)^{2/3}\sqrt{v} + 150\sqrt{35}\,u^{1/3})/(u^{1/3}\sqrt{v})\right]^{1/2}. \end{aligned}$$

In both of the above expressions,

$$\begin{aligned} u &= -450 + 210\sqrt{35}\,i, \\ v &= \frac{-45\,u^{1/3} + 480\cdot 2^{1/3} + 4\,(2u)^{2/3}}{u^{1/3}}. \end{aligned}$$

Since

$$z + \frac{1}{z} - \bar{\alpha} = \frac{1}{\bar{z}} + \bar{z} - \bar{\alpha} = \overline{z + \frac{1}{z} - \alpha} = \frac{1}{\bar{z}}\overline{(z^2 - \alpha z + 1)}.$$

We have

$$Q(z) = 35(z^2 - \alpha z + 1)\overline{(z^2 - \alpha z + 1)}(z^2 - \beta z + 1)\overline{(z^2 - \beta z + 1)}.$$

Let

$$s = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2} \quad \text{and} \quad t = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}. \tag{5}$$

Then

$$\begin{aligned} z^4Q(z) &= 35z^4(z-s)(z-\frac{1}{s})\overline{(z-s)(z-\frac{1}{s})}(z-t)(z-\frac{1}{t})\overline{(z-t)(z-\frac{1}{t})} \\ &= 35(z-s)(z-\frac{1}{s})(1-\bar{s}z)(1-\frac{1}{\bar{s}}z)(z-t)(z-\frac{1}{t})(1-\bar{t}z)(1-\frac{1}{\bar{t}}z) \\ &= 35(z-s)(z-\frac{1}{s})(z-\frac{1}{\bar{s}})(z-\bar{s})(z-t)(z-\frac{1}{t})(z-\frac{1}{\bar{t}})(z-\bar{t}) \\ &= 35\Big((z-s)(z-\bar{s})(z-t)(z-\bar{t})\Big)\Big((z-\frac{1}{s})(z-\frac{1}{\bar{s}})(z-\frac{1}{t})(z-\frac{1}{\bar{t}})\Big) \end{aligned}.$$

If

$$(z-s)(z-\bar{s})(z-t)(z-\bar{t}) = b_0 + b_1z + b_2z^2 + b_3z^3 + z^4,$$

then

$$(z-\frac{1}{s})(z-\frac{1}{\bar{s}})(z-\frac{1}{t})(z-\frac{1}{\bar{t}})=\frac{1}{b_0}(b_0z^4+b_1z^3+b_2z^2+b_3z+1).$$

Therefore we can set

$$\begin{aligned}L(z)&=\sqrt{\frac{35}{b_0}}(z-s)(z-\bar{s})(z-t)(z-\bar{t})\\&=a_0+a_1z+a_2z^2+a_3z^3+a_4z^4\\&:=\sqrt{35}|s||t|-2\sqrt{35}(\frac{\Re s}{|s|}|t|+\frac{\Re t}{|t|}|s|)z+\sqrt{35}\frac{|s|^2+4\Re s\,\Re t+|t|^2}{|s||t|}z^2\\&\quad-2\sqrt{35}\frac{\Re s+\Re t}{|s||t|}z^3+\sqrt{35}\frac{1}{|s||t|}z^4.\end{aligned}$$

Here $\Re s$ means the real part of s, and $\Re t$ the real part of t.

It does not matter if we choose $\bar{\alpha}$ instead of α. If we did, then $\bar{s}$ switches its role with s, and $L(z)$ does not change. If we choose both of the $\pm$'s in (5) to be $+$, it will give the "minimum phase" orthonormal scaling function in [1]. If we choose the $\pm$ to be $+$ for s and $-$ for t, it will give the "least asymmetric" orthonormal scaling function in [1]. The other two choices will give the reflections of the above two choices. Therefore, ignoring the reflections, there are two different sets of feasible coefficients a_k.

Once we have the values of a_k, the corresponding scaling coefficients c_k are determined by

$$(c_0,c_1,\ldots,c_9)=\frac{1}{128\sqrt{2}}(1,5,10,10,5,1)*(a_0,a_1,a_2,a_3,a_4).$$

REFERENCES

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