On the Exact Values of Orthonormal Scaling Coefficients of Length 8 and 10*

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Abstract. We show the exact values of the scaling coefficients of length eight and ten for Daubechies' orthonormal scaling functions.

In a recent article by Calderbank, Daubechies, Sweldens and Yeo [1], they wrote "... orthonormal filters with an assigned number N ($N \geq 2$) of vanishing moments ... (these) coefficients are real numbers which can be computed with high precision, but for which we do not even have a closed form expression if $N > 3$." Well, it happens that we know the closed form expressions for $N = 4$ and $N = 5$. Here we describe briefly the derivation of the exact values. For more details, please read the *Ten Lectures* [2] and the technical report by Shann and Yen [3].

Let $\{c_0, c_1, \ldots, c_{2p-1}\}$ be the orthonormal filters with an assigned number p of vanishing moments. Define the frequency response function

$$m_0(\xi) = \frac{1}{2}\sum_{k=0}^{2p-1} c_k e^{-ik\xi}.$$

Let $z = e^{-i\xi}$. Take the z-transform of the frequency response function $m_0(\xi)$, we have

$$H(z) = \frac{1}{2}\sum_{k=0}^{2p-1} c_k z^k = C(1+z)^p L(z), \tag{1}$$

where $L(z)$ is a polynomial of degree $p-1$:

$$L(z) = \sum_{k=0}^{p-1} a_k z^k$$

and C is determined such that $H(1) = 1$. Define

$$Q(z) = \sum_{k=0}^{p-1} \binom{p-1+k}{k} (\frac{2-z-z^{-1}}{4})^k.$$

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Note that $z^{p-1}Q(z)$ is a polynomial of degree $2p-2$ with symmetric coefficients, and it is the z-transform of the polynomial $P_N(y)$ in (6.1.12) of [2]. We will have to determine $L(z)$ such that $z^{p-1}Q(z) = L(z)\overline{L(z)}$.

Since $|z|=1$ and we restrict ourselves to the real valued a_k,

$$\overline{L(z)} = \sum_{k=0}^{p-1} a_k \bar{z}^k = \sum_{k=0}^{p-1} a_k z^{-k} = \frac{1}{z^{p-1}} \sum_{k=0}^{p-1} a_k z^{p-1-k}.$$

We can determine a_k by comparing the coefficients of

$$z^{p-1}Q(z) = (a_0 + a_1 z + \cdots + a_{p-1}z^{p-1})(a_0 z^{p-1} + \cdots + a_{p-2}z + a_{p-1}).$$

For the case of $p=4$,

$$z^3 Q(z) = -5 + 40z - 131z^2 + 208z^3 - 131z^4 + 40z^5 - 5z^6$$

and $C = \frac{1}{64}$ in (1). We would like to solve the four unknowns a_0, …, a_3 by the comparison of coefficients. It amounts to the solution of a system of four quadratic equations. The trick is that we can replace two of the quadratic equations by the linear equations $a_0 + a_1 + a_2 + a_3 = 4$ and $a_0 - a_1 + a_2 - a_3 = 2\sqrt{140}$. Then, we can reduce the above system of equations to a parameter α:

$$a_0 = \alpha \quad a_1 = 2 - \sqrt{140} + \frac{5}{\alpha} \quad a_2 = 2 + \sqrt{140} - \alpha \quad a_3 = -\frac{5}{\alpha}$$

where α is a root of the following polynomial:

$$\alpha^4 - 2(1+\sqrt{35})\alpha^3 + 40\alpha^2 + 10(1-\sqrt{35})\alpha + 25 = 0.$$

There is an algebraic formula for the solution of polynomials of degree 4, and a computer algebra system knows how to do it.

There are a pair of conjugate complex roots and two different real roots:

$$\alpha_1 = \frac{1}{2} + \frac{1}{2}\sqrt{35} + \frac{1}{6}\sqrt{\frac{3v}{u^{1/3}}} + \frac{1}{6}\sqrt{3s}$$
$$\alpha_2 = \frac{1}{2} + \frac{1}{2}\sqrt{35} + \frac{1}{6}\sqrt{\frac{3v}{u^{1/3}}} - \frac{1}{6}\sqrt{3s}$$

Here

$$u = 700 + 210\sqrt{15},$$

$$v = 28\,u^{1/3} - 70\cdot 2^{1/3} + (2u)^{2/3} + 6\,u^{1/3}\sqrt{35},$$

$$s = \frac{56\,u^{1/3}\sqrt{v} + 70\cdot 2^{1/3}\sqrt{v} - (2u)^{2/3}\sqrt{v} + 12u^{1/3}\sqrt{35v} + 336\sqrt{3u} + 48\sqrt{105u}}{u^{1/3}\sqrt{v}}.$$

Once we have the values of a_k, the scaling coefficients c_k are determined by

$$(c_0, c_1, \ldots, c_7) = \frac{1}{32}(1,4,6,4,1) * (a_0, a_1, a_2, a_3),$$

where $*$ means the discrete convolution. If we choose $\alpha = \alpha_1$, we have the "minimum phase" orthonormal scaling function. If we choose $\alpha = \alpha_2$, we have the "least asymmetric" orthonormal scaling function.

For the case of $p = 5$,

$$z^4 Q(z) = 35 - 350z^1 + 1520z^2 - 3650z^3 + 5018z^4 - 3650z^5 + 1520z^6 - 350z^7 + 35z^8$$

and $C = \frac{1}{256\sqrt{2}}$ in (1). The trick is to write

$$\begin{aligned} Q(z) &= 35(z+\frac{1}{z})^4 - 350(z+\frac{1}{z})^3 + 1380(z+\frac{1}{z})^2 - 2600(z+\frac{1}{z}) + 2048 \\ &= 35(z^2-\alpha z+1)\overline{(z^2-\alpha z+1)}(z^2-\beta z+1)\overline{(z^2-\beta z+1)}. \end{aligned}$$

Again, a computer algebra system fits in here and solves α and β:

$$\begin{aligned} \alpha = {} & \frac{5}{2} + \frac{1}{70}\sqrt{35v} \\ & - \frac{1}{70}\sqrt{35}\left[(-90\,u^{1/3}\sqrt{v} - 480\cdot 2^{1/3}\sqrt{v} - 4\,(2u)^{2/3}\sqrt{v} - 150\sqrt{35}\,u^{1/3})/(u^{1/3}\sqrt{v})\right]^{1/2}. \end{aligned}$$

and

$$\begin{aligned} \beta = {} & \frac{5}{2} + \frac{1}{70}\sqrt{35v} \\ & - \frac{1}{70}\sqrt{35}\left[(-90\,u^{1/3}\sqrt{v} - 480\cdot 2^{1/3}\sqrt{v} - 4\,(2u)^{2/3}\sqrt{v} + 150\sqrt{35}\,u^{1/3})/(u^{1/3}\sqrt{v})\right]^{1/2}. \end{aligned}$$

Here

$$u = -450 + 210\sqrt{35}\,i,$$

$$v = \frac{-45\,u^{1/3} + 480\cdot 2^{1/3} + 4\,(2u)^{2/3}}{u^{1/3}}.$$

Let

$$s = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2} \quad \text{and} \quad t = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}. \tag{2}$$

Then

$$z^4 Q(z) = 35\Big((z-s)(z-\bar{s})(z-t)(z-\bar{t})\Big)\Big((z-\frac{1}{s})(z-\frac{1}{\bar{s}})(z-\frac{1}{t})(z-\frac{1}{\bar{t}})\Big).$$

Therefore we can set

$$L(z) = \frac{\sqrt{35}}{|s|\,|t|}(z-s)(z-\bar{s})(z-t)(z-\bar{t}).$$

Thus

$$(a_0, a_1, a_2, a_3, a_4) = \frac{\sqrt{35}}{|s|\,|t|}(|s|^2|t|^2, -2(|t|^2\Re s + |s|^2\Re t), |s|^2 + 4\Re s\,\Re t + |t|^2, -2(\Re s + \Re t), 1).$$

Here $\Re s$ means the real part of s, and $\Re t$ the real part of t.

Once we have the values of a_k, the corresponding scaling coefficients c_k are determined by

$$(c_0, c_1, \ldots, c_9) = \frac{1}{128\sqrt{2}}(1, 5, 10, 10, 5, 1) * (a_0, a_1, a_2, a_3, a_4).$$

It does not matter if we choose $\bar{\alpha}$ instead of α. If α and $\bar{\alpha}$ switch their roles, then s and $\bar{s}$ switch their roles and $L(z)$ does not change. If we choose both of the $\pm$'s in (2) to be $+$, we have the "minimum phase" orthonormal scaling function. If we choose the $\pm$ to be $+$ for s and $-$ for t, we have the "least asymmetric" orthonormal scaling function. The other two choices will give the reflections of the above two choices.

REFERENCES

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