國立中央大學

數學研究所

碩士論文

一套提昇凌波函數
逼近能力與平滑度的方法

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數學研究所曾正男君所提之論文

一套提昇基波函數逼近能力與平滑度的方法

（題目）

係由本人指導撰述，同意提付審查。

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（題　　　目）

經本委員會審議，認為符合碩士資格標準。

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論文摘要

在此論文中, 我們提出一套提昇自格函數與凌波函數逼近能力與平滑度的方法. 將此提昇法搭配 Sweldens 的拉舉法 (lifting scheme), 我們可以從給定的雙正交凌波系統中, 很快速的造出無窮多組新的雙正交凌波系統.

特別地, 利用此方法我們可將 Daubechies 正交自格與正交凌波函數, 提昇爲一套的交互正交的自格與凌波函數. 其中自格函數的導函數與其自己的平移幾乎正交, 而凌波函數的導函數與其自己的平移正交, 而且與自格函數的導函數也正交. 這種微分後正交的特性對於微分方程的數值解特別有幫助. 在此基底下所得到的剛性矩陣, 可藉著微分後正交以及凌波函數快速基底轉換這兩種特性, 同時享有稀疏 (sparse) 而又良性 (well-conditioned) 的好處. 理論上, 用凌波函數來做微分方程數值解的基底, 不只有多重網格法 (multigrid method) 層層互換的特性, 並且我們可以將高頻的部分算得清楚, 而不是省略.

第一章的內容是介紹凌波方程的理論. 我們希望採用的觀點是給定一組係數, 當此係數滿足某些條件時, 此係數能生成一自格函數, 而此自格函數能生成一多重解分析, 再由多重解分析生成對應的凌波函數. 最後從這一組自格凌波函數, 我們又能找到一組對偶的自格凌波函數, 而完成整套凌波方程的架構.

我們盡量使用一致的符號及證明方法, 來推導各家不同形式的凌波方程. 例如 Daubechies 的正交凌波方程， 崔王的半正交凌波方程, 與 Cohen-Daubechies-Feauveau 的交互正交凌波方程. 這一章的目的是方便讀者瞭解各家的精神與各家的差異性.

第二章共分四節. 第一節説明利用疊積提昇後的自格函數仍是自格函數. 第二節説明提昇後的自格函數仍展出多重解分析. 第三節説明不可能存在一正交的自格函數經提昇後仍爲正交自格函數. 第四節主要在討論提昇後的自格函數與其對偶函數的關係. 對於一般交互正交的自格與凌波函數, 只要其平滑度足夠, 我們都能用此提昇法造出另一套交互正交的自格與凌波函數.

ABSTRACT

A SCHEME TO ELEVATE THE DEGREE AND REGULARITY OF BIORTHOGONAL WAVELETS

We will demonstrate a scheme that will elevate orthonormal or biorthogonal wavelets to a new pair of biorthogonal wavelets. The approximation degree (the number of vanishing moments) and the regularity of the elevated wavelet will be higher than the original wavelet.

If we start with B-splines and cooperate with Sweldens' *lifting scheme*, the elevation scheme can reproduce the compactly supported biorthogonal wavelets of Cohen, Daubechies and Feauveau. There is a simple algorithm to elevate from old filter coefficients to new filter coefficients.

If we start with a pair of orthonormal scaling function and wavelet, the elevated scaling function and wavelet are differentiable. The derivative of the elevated scaling function is orthonormal to all but the nearest two of its integer translations. The derivative of the elevated wavelet is orthonormal to all of its integer translations, and it is orthogonal to the derivative of the elevated scaling function. Both the elevated and original scaling functions and wavelets have the structure of multiresolutional analysis, this property enables the fast algorithm of decomposition and reconstruction. These features may be valuable for the solution of differential equations.

PART ONE

第一章

凌波理論概述

我們將在這一章介紹基本的凌波理論, 我們希望用比較建構性的方法來介紹凌波理論. 首先, 我們選定一組有限長的係數, 當此係數滿足某些條件時, 此係數會對應生成一自格函數 (scaling function), 且此自格函數可以展成一多重解分析 (multiresolution analysis), 再由此多重解分析我們建構出所需要的凌波函數 (wavelet).

第一節我們介紹自格函數的基本性質, 從第二節以後我們開始介紹凌波函數. 在二, 三, 四節中我們分別介紹正交凌波函數, 半正交凌波函數以及雙正交凌波函數. 特別在第三節我們用不同於原著者的證法來介紹半正交凌波函數, 以方便讀者比較上述三種凌波函數彼此的關係與差異.

第一節 自格函數

首先我們定義何爲自格函數. 若一平方可積函數 $\phi(x)$ 滿足下列關係

$$\phi(x)=\sum c_n\phi(2x-n), \tag{1.1}$$

則我們稱函數 $\phi(x)$ 滿足自格等式, 且 (c_n) 稱爲 $\phi(x)$ 的自格係數. 由於 $\phi(2x-n)$ 是 $\phi(x)$ 橫向壓縮一半再向右平移 $n/2$ 個單位, 從 (1.1)我們可以看出函數 $\phi(x)$ 被自己的壓縮和平移再建構回來, 故 (1.1)稱爲自格等式. 當我們說 $\phi(x)$ 爲一自格函數時, 我們特別指 $\phi(x-n)$ 的生成空間包含某種階數以內的多項式. 例如,

$$\sum\phi(x-n)=1.$$

由自格等式我們可以得到, 若函數 $\phi(x)\in L^1(\mathbb{R})$ 則

$$\hat{\phi}(\omega)=\frac{1}{2}\sum c_n\, e^{-i\frac{\omega}{2}n}\hat{\phi}(\frac{\omega}{2}).$$

我們定義一 2π 週期函數 $m_0^{\phi}(\omega)$ 爲 $\phi(x)$ 的特徵函數

$$m_0^{\phi}(\omega)=\frac{1}{2}\sum c_n\, e^{-i\omega n},$$

則

$$\hat{\phi}(\omega)=m_0^{\phi}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}). \tag{1.2}$$

當前後文不混淆的時候, 我們用 $m_0(\omega)$ 代替 $m_0^{\phi}(\omega)$. 由 (1.2) 我們知道, 若 $\hat{\phi}(0)\neq 0$, 自格係數滿足 $\sum c_n=2$. 在實用上我們只關心 $\hat{\phi}(0)\neq 0$ 這一種情形, 因爲只有在 $\hat{\phi}(0)\neq 0$ 時, $\phi(x-n)$ 才有作爲一組 Riese basis 的可能, 且習慣上我們調整 $\phi(x)$ 的規格使得 $\hat{\phi}(0)=1$. 根據 (1.1) 得到

$$\hat{\phi}(\omega)=\prod_{j=1}^{\infty}m_0(2^{-j}\omega).$$

我們可以看到 $\sum c_n=2$ 這個條件在這裡是必要的, 此條件使得 $m_0^{\phi}(0)=1$, 以至於上述的無窮乘積在任意一個實數的緊緻子集內均勻收斂.

到現在爲止我們還不知道給定任意一組自格係數 (c_n), 是否眞正存在一自格函數 $\phi(x)$ 滿足對應此自格係數的自格等式. 若存在 $\phi(x)$ 滿足自格等式, 這樣的 $\phi(x)$ 是否屬於某一個 $L^p(\mathbb{R})$ 空間? 以下的定理將告訴我們, 只要給定一有限長的自格係數, 如 $c_0,\ldots,c_N$, 必存在唯一的廣義函數 (distribution) $\phi(x)$, 滿足 (c_n) 所決定的自格等式; 且 $\phi(x)$ 是有限涵蓋, 其函蓋落在 $[0,N]$ 區間內. [7]

Theorem 1.1. *Let* $c_0,\ldots,c_N\in\mathbb{R}$ *with* $\sum c_n=2$. *Then there is a unique compactly supported distribution* ϕ *which satisfies* (1.1) *with* $\hat{\phi}(0)=1$. *The support of* ϕ *is contained in* $[0,N]$.

在 Lars F. Villemoes 的一篇論文裡 [9] 提到由自格係數 (c_n) 判別自格函數 $\phi(x)$ 是否屬於 $L^2(\mathbb{R})$ 的方法. 給定一個 2π 周期函數 $m_0(\omega)$, 我們定義運算元 A 如下:

$$Af(\omega)=\left|m_0(\frac{\omega}{2})\right|^2 f(\frac{\omega}{2})+\left|m_0(\frac{\omega}{2}+\pi)\right|^2 f(\frac{\omega}{2}+\pi).$$

假設 $\phi(x)\in L^2(\mathbb{R})$. 令

$$\gamma_n=\int\phi(x)\phi(x-n)\,dx,$$

定義

$$\Gamma(\omega) = \sum |\hat{\phi}(\omega + 2\pi n)|^2. \tag{1.3}$$

由 Plancherel's theorem 得知

$$\Gamma(\omega) = \sum \gamma_n e^{-i\omega n}.$$

由於 $m_0(\omega)$ 是 2π 週期函數, 加上 (1.2) 我們有

$$\begin{aligned}\Gamma(2\omega) &= \sum |m_0(\omega + \pi n)|^2 |\hat{\phi}(\omega + \pi n)|^2 \\ &= |m_0(\omega)|^2 \Gamma(\omega) + |m_0(\omega + \pi)|^2 \Gamma(\omega + \pi),\end{aligned}$$

亦即, $A\Gamma = \Gamma$.

Theorem 1.2. *We have* $\phi(x) \in L^2(\mathbb{R})$ *and* $\hat{\phi}(0) \neq 0$ *if and only if there exists a real nonnegative trigonometric polynomial* $\Gamma(\omega)$ *satisfying* $\Gamma(0) > 0$ *and* $A\Gamma = \Gamma$.

由以上定理我們知道, 只要數列 (γ_n) 存在且 $\sum \gamma_n > 0$ 則 $\phi(x) \in L^2(\mathbb{R})$. 由於自格函數有別於一般的函數, 我們有一套特別的方法可求得數列 (γ_n). 略述如下: 從 (γ_n) 的定義可知, 數列 (γ_n) 是一組對稱的數列, 亦即, $\gamma_n = \gamma_{-n}$. 當自格係數 (c_n) 是有限長的時候, $\phi(x)$ 是有限涵蓋. 因此, 只有有限長的 (γ_n) 是非零的. 由於 $\phi(x)$ 滿足自格等式 (1.1) , 我們有

$$\begin{aligned}\gamma_n &= \langle \phi(x), \phi(x-n) \rangle \\ &= \langle \sum c_i \phi(2x - i), \sum c_j \phi(2x - 2n - j) \rangle \\ &= \frac{1}{2} \sum_{i,j} c_i c_j \gamma_{2n+j-i} \\ &= \frac{1}{2} \sum_{i,j} c_i c_{i+j-2n} \gamma_j\end{aligned}$$

因此我們得到一組 (γ_n) 的聯立方程式, 而這個問題變成解一個固有值是 1 的固有向量問題.

在實做上我們會碰上兩種失敗的情形. 第一種情形是, 給定的 (c_n) 所造成的矩陣無法解出固有值是 1 的固有向量; 第二種情形是, 解出來的固有向量的和等於零.

當第一種情形發生時, $\phi(x)$ 當然不會是平方可積函數, 因爲我們求不出 γ_0 這一個量. 第二種情形是因爲我們假設 $\hat{\phi}(0) \neq 0$, 我們必需有 $\sum \gamma_n \neq 0$ 的結果. 特別的, $\sum \gamma_n \neq 0$ 這一個特性可推導到 $\Gamma(0) > 0$, 且 $\Gamma(\omega)$ 滿足 $A\Gamma = \Gamma$. 所以, 當 $\sum \gamma_n = 0$ 時, 所求得的自格函數 $\phi(x)$ 也不屬於 $L^2(\mathbb{R})$, 或所求得的自格函數有可能屬於 $L^2(\mathbb{R})$ 但 $\hat{\phi}(0) = 0$. 從這裡我們不難看出, 要造一個平方可積的自格函數並不困難.

當我們說 $\phi(x)$ 是自格函數時, 我們強調 $\phi(x-n)$ 的生成集包含了某種階數以下的多項式. $\phi(x-n)$ 的逼近能力, 事實上是由自格係數 (c_n) 所決定的. 我們有以下性質:

Proposition 1.1. *Let* $\phi(x)$ *be defined as in* (1.1)*. If the scaling coefficients* (c_n) *satisfy*

$$\sum_{n\in\mathbb{Z}} (-1)^n n^m c_n = 0, \tag{1.4}$$

for $0 \leq m \leq p$, *and the scaling function satisfies*

$$\sum_{n\in\mathbb{Z}} \phi(x-n) = 1, \quad \forall x \in \mathbb{R},$$

then $\sum\limits_{n\in\mathbb{Z}} n^m \phi(x-n)$ *is a polynomial of degree* m.

$\int \phi(x)\,dx = 1$ 的性質 (其實只要 $\int \phi(x)\,dx \neq 0$ 就可以了) 加上 (1.1) 可推得 $\sum c_n = 2$. 反之, 在下面的性質中, 我們將證明 $\sum c_n = 2$, (1.1) 以及 $\int \phi(x)\,dx = 1$ 這三個條件, 也可以推得 $\sum \phi(x-n) = 1$. 總括而言, 我們可將上述性質修改爲以下的敘述.

Proposition 1.2. *Let* $\phi(x)$ *be defined as in* (1.1) *and* $\sum c_n = 2$*. If the scaling coefficients* (c_n) *satisfy*

$$\sum_{n\in\mathbb{Z}} (-1)^n n^m c_n = 0 \tag{1.5}$$

for integer $0 \leq m \leq p$, *then* $\sum\limits_{n\in\mathbb{Z}} n^m \phi(x-n)$ *is a polynomial of degree* m.

我們可以避開傅氏分析來證明此一性質.

Proof: Let $f(x) = \sum \phi(x-n)$. By (1.1) we have

$$\begin{aligned} f(x) &= \sum_n \phi(x-n) \\ &= \sum_n \sum_k c_k \phi(2x-2n-k) \\ &= \sum_k c_k \sum_n \phi(2x-2n-k) \quad \left(\text{finitely many } c_k \neq 0\right) \\ &= \sum_k c_{2k} \sum_n \phi(2x-2n) + \sum_k c_{2k-1} \sum_n \phi(2x-2n-1) \\ &= \sum_n \phi(2x-n) \quad \left(\text{ since } \sum c_k = 2 \text{ and } \sum (-1)^k c_k = 0 \right) \\ &= f(2x). \end{aligned}$$

由 $f(x) = f(2x)$ 我們知道 $f(x)$ 爲一常數函數, 或分片的常數函數且只有在 0 那一點有斷點. 但我們從 $f(x)$ 的定義可以看到, 若 $f(x)$ 在 0 那一點上有斷點, 則 $f(x)$ 在每一個整數點上都有斷點. 這是一個矛盾的結果, 所以 $f(x)$ 是一個常數函數. 我們說 $f(x) = C$, 亦即 $\sum \phi(x-n) = C$. 再來我們要看看常數 C 到底會等於多少.

若 C 等於 0, 則 $\phi(x-n)$ 是線性相關的, 所以不是一組基底. 在實作上, 我們希望我們所擁有的 $\phi(x-n)$ 不只是一組基底, 而且最好是一組 Riesz basis. 當 C 不等於 0 時, $\phi(x)$ 的規模便決定了常數 C 的大小. 我們將 $\sum \phi(x-n) = C$ 在 $[0,1]$ 區間上作積分, 則左式恰爲 $\int \phi(x)\,dx$. 當我們令 $\int \phi(x) = 1$ 時, 常數 C 等於 1.

我們證明了 $m = 0$ 的情形, 其餘 $m = 1, \ldots, p$ 的情形只要用強數學歸納法證明即可. 從 (1.5) 得知 m 大於 p 以後便沒有上述性質. 我們可以發現, 當 (1.5) 的 p 越大, $\phi(x-n)$ 的生成集便可包含階數越高的多項式. □

若 $\phi(x-n)$ 的生成集包含了 p 階以及 p 階以下的多項式, 則我們稱 $\phi(x)$ 具有 p 階的逼近度.

事實上, 當 $\hat{\phi}(0) = 0$ 時, 上述性質也會成立. 此時我們得到的常數函數爲零函數. 由於 $\sum \phi(x-n) = 0$, 所以 $\phi(x-n)$ 不是一組基底. 雖然 $\phi(x-n)$ 不是一組基底, 但 $\phi(x-n)$ 仍然包含 $x, x^2, \cdots$ 等多項式. 這雖然不是一個很好的性質但我們仍提出來供讀者參考.

第二節 正交的凌波函數

在這一節的開始, 我們先定義何爲多重解分析.

Definition 1.1. *A multiresolution analysis is a sequence* (V_j) *of subspaces of* $L^2(\mathbb{R})$ *such that*

(1) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$,
(2) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ *and* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(3) $f(x) \in V_j \iff f(2x) \in V_{j+1}$,
(4) $f(x) \in V_0 \iff f(x-n) \in V_0 \quad \forall n \in \mathbb{Z}$.

給定一自格函數 $\phi(x)$, 我們定義 V_j 爲

$$V_j = \operatorname{span}\{\sqrt{2^j}\phi(2^jx-n) : n \in \mathbb{Z}\}.$$

若 $\hat{\phi}(0) = 1$ 且 $\phi(x-n)$ 是一組 Riesz basis, 則 $\phi(x)$ 生成一多重解分析.

我們定義 W_j 爲 V_j 在 V_{j+1} 的正交補集, 即

$$V_{j+1} = V_j \oplus W_j.$$

如果 W_j 是被一個單一的函數 $\psi(x)$ 的平移所展成的, 即

$$W_j = \operatorname{span}\{\sqrt{2^j}\psi(2^jx-n) \mid n \in \mathbb{Z}\}, \tag{1.6}$$

而且 $\psi(x)$ 至少是指數遞減, 則我們稱 $\psi(x)$ 爲相對應於 $\phi(x)$ 的凌波函數. 我們可以由此多重解分析導出相對應的凌波函數.

當 $\phi(x-n)$ 是一組正交基底時, 我們有如下的性質:

$$\begin{aligned}
\delta_{k,0} &= \int \phi(x)\overline{\phi(x-k)}dx \\
&= \int \overline{\hat{\phi}(\omega)}\hat{\phi}(\omega)e^{i\omega k}d\omega \\
&= \int |\hat{\phi}(\omega)|^2 e^{ik\omega}d\omega \\
&= \int_0^{2\pi} [\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega+2\pi n)|^2] e^{ik\omega}d\omega.
\end{aligned}$$

因爲 k 是任意的, 所以

$$\sum |\hat{\phi}(\omega + 2\pi n)|^2 = (2\pi)^{-1} \quad \text{a.e.} \tag{1.7}$$

我們將 (1.2) 代入, 則

$$\sum |m_0(\omega + \pi n)|^2 |\hat{\phi}(\omega + \pi n)|^2 = (2\pi)^{-1} \quad \text{a.e.};$$

我們再將奇數項及偶數項拆開, 即可得到

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1 \quad \text{a.e.} \tag{1.8}$$

這是正交自格函數的一個重要的性質.

我們現在開始推導 $\psi(x)$. 令 $f \in W_0$. 根據 W_0 的定義我們知道 $f \in V_1$ 且 $f \perp V_0$. 因爲 $f \in V_1$, 我們有

$$f = \sum f_n \phi(2x - n),$$

這裡的 f_n 是 $f_n = \langle f(x), \phi(2x - n) \rangle$. 我們可以得到

$$\hat{f}(\omega) = \frac{1}{2} \sum f_n e^{-in\frac{\omega}{2}} \hat{\phi}(\frac{\omega}{2}) = m_f(\frac{\omega}{2}) \hat{\phi}(\frac{\omega}{2}), \tag{1.9}$$

其中 $m_f(\omega)$ 被定義爲

$$m_f(\omega) = \frac{1}{2} \sum f_n e^{-in\omega}; \tag{1.10}$$

m_f 是一個 2π 週期函數, 且 (1.10) 是點收斂. $f \perp V_0$ 這一個條件可以推得 $f \perp \phi_{0,k}$ 對所有的 k 都成立, 即

$$\int \hat{f}(\omega) \overline{\hat{\phi}(\omega)} e^{-ik\omega} \, d\omega = 0$$

或者

$$\int_0^{2\pi} [\sum_{n \in \mathbb{Z}} \hat{f}(\omega + 2\pi n) \overline{\hat{\phi}(\omega + 2\pi n)}] e^{-ik\omega} \, d\omega = 0 \quad \forall k.$$

因此, 我們有

$$\sum \hat{f}(\omega + 2\pi n) \overline{\hat{\phi}(\omega + 2\pi n)} = 0 \tag{1.11} \text{ a.e.}$$

並且上述的式子在 $L^1([-\pi, \pi])$ 上是絕對收斂的. 我們將 (1.2) 和 (1.9) 代入, 並且一樣的將奇數項與偶數項拆開, 再利用 (1.7) 可以得到

$$m_f(\omega) \overline{m_0(\omega)} + m_f(\omega + \pi) \overline{m_0(\omega + \pi)} = 0 \quad \text{a.e.} \tag{1.12}$$

根據 (1.8), $m_0(\omega)$ 和 $m_0(\omega+\pi)$ 不會同時爲 0, 所以存在一 2π 週期函數 $\lambda(\omega)$ 使得

$$m_f(\omega)=\lambda(\omega)\overline{m_0(\omega+\pi)} \quad \text{a.e.} \tag{1.13}$$

由於 (1.12) 我們有

$$\lambda(\omega)+\lambda(\omega+\pi)=0 \quad \text{a.e.} \tag{1.14}$$

摹仿 Daubechies [5] 的證法, $\lambda(\omega)$ 可以寫成

$$\lambda(\omega)=e^{i\omega}\nu(2\omega), \tag{1.15}$$

這裡的 ν 也是一個 2π 的週期函數. 將 (1.15) 以及 (1.13) 代入 (1.9) 得到

$$\hat{f}(\omega)=e^{i\omega/2}\overline{m_0(\frac{\omega}{2}+\pi)}\nu(\omega)\hat{\phi}(\frac{\omega}{2}). \tag{1.16}$$

我們將證明正交凌波函數滿足如下形式

$$\hat{\psi}(\omega)=e^{i\frac{\omega}{2}}\overline{m_0(\frac{\omega}{2}+\pi)}\hat{\phi}(\frac{\omega}{2}). \tag{1.17}$$

亦即 $\hat{\psi}(\omega)=m_1(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2})$, 其中

$$m_1(\omega)=e^{i\omega}\overline{m_0(\omega+\pi)}.$$

如果 $m_1(\omega)=\frac{1}{2}\sum d_n e^{-i\omega n}$, 則

$$\psi(x)=\sum d_n\phi(2x-n).$$

我們暫時不理會收斂性的問題, (1.16) 可以寫成

$$\hat{f}(\omega)=\left(\sum \nu_k e^{-ik\omega}\right)\hat{\psi}(\omega). \tag{1.18}$$

亦即

$$f(x)=\sum \nu_k\psi(x-k).$$

Proposition 1.3. *Let $\phi(x)$ be a scaling function such that $\phi(x-n)$ is an orthonormal basis of V_0. Define the wavelet $\psi(x)$ as in (1.17), then $\psi(x-n)$ is an orthonormal basis of W_0.*

Proof: 首先, 根據 m_0 和 $\hat{\phi}$ 的性質, 再由 (1.17) 我們可以得到 $\psi(x) \in V_1$ 且 $\psi(x) \perp V_0$ 如果 $f(x-k)$ 是 W_0 的一個正則集合,則

$$\begin{aligned}\delta_{0,k} &= \int |\hat{f}(\omega)|^2 e^{i\omega k}\, d\omega \\ &= \int |m_0(\frac{\omega}{2}+\pi)\hat{\phi}(\frac{\omega}{2})\nu(\omega)|^2 e^{i\omega k}\, d\omega \\ &= \int_0^{2\pi} \sum |m_0(\frac{\omega}{2}+(n+1)\pi)\hat{\phi}(\frac{\omega}{2}+n\pi)\nu(\omega)|^2 e^{i\omega k}\, d\omega.\end{aligned}$$

由於 k 是任意的整數, 我們有

$$\sum |m_0(\frac{\omega}{2}+(n+1)\pi)\hat{\phi}(\frac{\omega}{2}+n\pi)\nu(\omega)|^2 = (2\pi)^{-1}.$$

再利用 (1.8) 以及 (1.7), 我們可以得到

$$|\nu(\omega)|^2 \left\{ \sum |\hat{\phi}(\frac{\omega}{2}+n\pi)|^2 - \sum |m_0(\frac{\omega}{2}+n\pi)|^2 |\hat{\phi}(\frac{\omega}{2}+n\pi)|^2 \right\} = (2\pi)^{-1}.$$

可見 $\nu(\omega)$ 必然是 ± 1. 所以, $f(x) = \pm\psi(x)$. 也就是說, 當我們決定了一個由正交自格函數所展出的多重解分析的時候, 一般而言, 相對應的凌波函數未必是唯一決定的, 但正交的凌波函數卻是唯一 (除了 ± 號以外) 被決定的.

再來我們要檢驗 $\psi_{0,k}$ 是 W_0 的基底. 對於任意的 $f \in W_0$, 回顧 (1.18) 我們有 $\hat{f}(\omega) = \gamma(\omega)\hat{\psi}(\omega)$, 其中 $\gamma(\omega) = \sum \gamma_n e^{-i\omega n}$ 且

$$f(x) = \sum \gamma_n \psi(x-n).$$

因此, 只須證明 $\sum |\gamma_n|^2 < \infty$. 因爲

$$\int_0^{2\pi} |\gamma(\omega)|^2 d\omega = 2\int_0^{\pi} |\lambda(\omega)|^2 d\omega.$$

根據 (1.10) 我們得到

$$\int_0^{2\pi} |m_f(\omega)|^2 d\omega = \pi \sum |f_n|^2 = \pi \|f\|^2 < \infty.$$

再另一方面, 根據 (1.13) 我們有

$$\begin{aligned}\int_0^{2\pi} |m_f(\omega)|^2 d\omega &= \int_0^{2\pi} |\lambda(\omega)|^2 |m_0(\omega+\pi)|^2 d\omega \\ &= \int_0^{\pi} |\lambda(\omega)|^2 [|m_0(\omega+\pi)|^2 + |m_0(\omega)|^2] d\omega \qquad \text{By (1.14)} \\ &= \int_0^{\pi} |\lambda(\omega)|^2\, d\omega. \qquad \text{By (1.8)}\end{aligned}$$

由於 $\int_0^{2\pi} |\gamma(\omega)|^2 d\omega = 2\pi\|f\|^2 < \infty$, 得到 (γ_n) 是平方可加的. 從以上我們得知 $\psi(x-n)$ 的確是 W_0 的基底. $\square$

在一般的情形, 自格函數不會是正交的. 對於這種不是正交的自格函數, 我們介紹兩種較著名的製造凌波函數的方法: 第一種是所謂的半正交凌波函數, 另一種是所謂的雙正交凌波函數. 以下兩節分別來介紹這兩種凌波函數的造法.

第三節 半正交凌波函數

首先我們先介紹何爲半正交凌波函數. 延用上一節的符號, 若

$$V_{j+1} = V_j \oplus W_j.$$

則我們稱那一個展出 W_0 的函數 $\psi(x)$ 爲相對應於 $\phi(x)$ 的半正交凌波函數. 這裡我們要注意的是 $\psi(x-n)$ 與 $\phi(x-n)$ 是正交的, 但 $\psi(x-n)$ 本身未必正交.

根據半正交性, 若 $\psi(x-n)$ 是 W_0 的一組基底, 則對於所有的整數 n, 如同正交凌波函數的情況, $\hat{\psi}(\omega) = m_1(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2})$. 我們有

$$\begin{aligned}
\langle \phi(x), \psi(x-n) \rangle &= \int \hat{\phi}(\omega)\overline{\hat{\psi}(\omega)} e^{i\omega n}\, d\omega \\
&= \int e^{i\omega n} m_0(\frac{\omega}{2})\overline{m_1(\frac{\omega}{2})} |\hat{\phi}(\frac{\omega}{2})|^2\, d\omega \\
&= \int_0^{4\pi} e^{i\omega n} m_0(\frac{\omega}{2})\overline{m_1(\frac{\omega}{2})} \sum |\hat{\phi}(\frac{\omega}{2} + 2\pi k)|^2\, d\omega \\
&= \int_0^{2\pi} e^{i\omega n} \Big(m_0(\frac{\omega}{2})\overline{m_1(\frac{\omega}{2})} \sum |\hat{\phi}(\frac{\omega}{2} + 2\pi k)|^2 \\
&\quad + m_0(\frac{\omega}{2} + \pi)\overline{m_1(\frac{\omega}{2} + \pi)} \sum |\hat{\phi}(\frac{\omega}{2} + 2\pi k + \pi)|^2 \Big)\, d\omega.
\end{aligned}$$

同樣的道理, 因爲 n 是任意的, 所以

$$m_0(\omega)\overline{m_1(\omega)}\Gamma(\omega) + m_0(\omega+\pi)\overline{m_1(\omega+\pi)}\Gamma(\omega+\pi) = 0 \quad \text{a.e.,} \tag{1.19}$$

這裡的 $\Gamma(\omega)$ 如 (1.3) 所定義. 從 (1.19) 我們可以得到 $\psi(x)$ 的特徵函數 $m_1(\omega)$ 可寫成

$$m_1(\omega) = e^{-i\omega}\overline{m_0(\omega+\pi)}\Gamma(\omega+\pi)s(2\omega) \quad \text{for some } s \in \mathcal{A}. \tag{1.20}$$

這裡的集合 $\mathcal{A}$ 是定義爲:

$$\mathcal{A} = \{S : S = \sum s_n e^{-i\omega n}, S(\omega) \neq 0 \text{ and } (s_n) \in \ell^1\}. \tag{1.21}$$

我們要證明 $\psi(x-n)$ 是 W_0 的一組基底.

Proposition 1.4. *Let* ϕ *be defined as in* (1.1) *and* $m_1(\omega)$ *be defined as above. If* $\Gamma(\omega)$ *is also in* $\mathcal{A}$*, then* ψ *is a semi-orthogonal wavelet of* ϕ*.*

Proof: 因爲 $\hat{\psi}(\omega) = m_1(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2})$, 加上 $m_0(\omega)$ 和 $\Gamma(\omega)$ 的週期性我們得到

$$\begin{aligned}\sum|\hat{\psi}(\omega+2\pi n)|^2 &= \sum|m_1(\frac{\omega}{2}+\pi n)\hat{\phi}(\frac{\omega}{2}+\pi n)|^2\\ &= \sum|e^{-i(\frac{\omega}{2}+\pi n)}m_0(\frac{\omega}{2}+\pi n+\pi)\\ &\qquad\times\Gamma(\frac{\omega}{2}+\pi n+\pi)s(\omega)\hat{\phi}(\frac{\omega}{2}+\pi n)|^2\\ &= |s(\omega)e^{-i\frac{\omega}{2}}|^2\Gamma(\frac{\omega}{2}+\pi)\Gamma(\frac{\omega}{2})\\ &\qquad\times\left(|m_0(\frac{\omega}{2}+\pi)|^2\Gamma(\frac{\omega}{2}+\pi)+|m_0(\frac{\omega}{2})|^2\Gamma(\frac{\omega}{2})\right).\end{aligned}$$

我們先觀察 $\Gamma(\omega)$. 因爲 $\Gamma(\omega)$ 定義爲

$$\Gamma(\omega) = \sum|\hat{\phi}(\omega+2\pi n)|^2.$$

我們將上式的奇數項與偶數項拆開, 利用 (1.2) 再重新組合得到

$$|m_0(\frac{\omega}{2}+\pi)|^2\Gamma(\frac{\omega}{2}+\pi)+|m_0(\frac{\omega}{2})|^2\Gamma(\frac{\omega}{2}) = \Gamma(\omega), \tag{1.22}$$

應用此結果我們得到

$$\sum|\hat{\psi}(\omega+2\pi n)|^2 = |s(\omega)e^{-i\frac{\omega}{2}}|^2\Gamma(\frac{\omega}{2}+\pi)\Gamma(\frac{\omega}{2})\Gamma(\omega).$$

因爲 $\phi(x-n)$ 是 V_0 的一組 Riesz basis, 所以 $\Gamma(\omega)$ 是有上下界的, 並且其下界是遠離零點的. 亦即存在常數 $0<A\le B<\infty$ 使得

$$A \le \sum|\hat{\phi}(\omega+2\pi n)|^2 \le B \quad \text{a.e.}.$$

因爲 $s(\omega)$ 屬於 $\mathcal{A}$, $s(\omega)$ 是一個 2π 週期連續函數, 所以 $s(\omega)$ 在 $[0,2\pi]$ 區間上有最大, 最小值. 我們令 $L=\min s(\omega)>0$ 及 $U=\max s(\omega)<\infty$, 因此我們有

$$A^3L^2 \le \sum|\hat{\psi}(\omega+2\pi n)|^2 \le B^3U^2 \quad \text{a.e.}.$$

所以 $\psi(x-n)$ 是 W_0 的一組基底.

再來我們要證明 $V_1 = V_0 + W_0$. 即存在兩數列 $(\tilde{c}_n)$ 和 $(\tilde{d}_n)$ 使得

$$\phi(2x-\ell) = \sum \{\tilde{c}_{\ell-2n}\phi(x-n) + \tilde{d}_{\ell-2n}\psi(x-n)\}. \tag{1.23}$$

我們希望數列 $(\tilde{c}_n)$ 和 $(\tilde{d}_n)$ 可以從 $m_0(\omega)$ 和 $m_1(\omega)$ 得到. 我們將 (1.23) 作傅利葉展開,

$$\begin{aligned}\frac{1}{2}\hat{\phi}(\frac{\omega}{2})e^{-i\ell\frac{\omega}{2}} &= \sum \tilde{c}_{\ell-2n}e^{-i\omega n}\hat{\phi}(\omega) + \sum \tilde{d}_{\ell-2n}e^{-i\omega n}\hat{\psi}(\omega) \\ &= \sum \tilde{c}_{\ell-2n}e^{-i\omega n}m_0(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}) + \sum \tilde{d}_{\ell-2n}e^{-i\omega n}m_1(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}).\end{aligned}$$

我們得到

$$\left(\sum \tilde{c}_{\ell-2n}e^{i(\ell-2n)\omega}\right)m_0(\omega) + \left(\sum \tilde{d}_{\ell-2n}e^{i(\ell-2n)\omega}\right)m_1(\omega) \equiv \frac{1}{2} \quad \forall \ell \in \mathbb{Z}. \tag{1.24}$$

這裡, 我們要新定義兩個特徵函數 $\widetilde{m}_0$ 和 $\widetilde{m}_1$ 爲

$$\widetilde{m}_0(\omega) = \sum \tilde{c}_n e^{-i\omega n} \quad \text{and} \quad \widetilde{m}_1(\omega) = \sum \tilde{d}_n e^{-i\omega n}.$$

同樣的, 我們將 (1.24) 的奇數項與偶數項拆開, 得到

$$(\widetilde{m}_0(\omega) + \widetilde{m}_0(\omega+\pi))\overline{m_0(\omega)} + (\widetilde{m}_1(\omega) + \widetilde{m}_1(\omega+\pi))\overline{m_1(\omega)} = 1 \quad \text{for even } \ell$$

以及

$$(\widetilde{m}_0(\omega) - \widetilde{m}_0(\omega+\pi))\overline{m_0(\omega)} + (\widetilde{m}_1(\omega) - \widetilde{m}_1(\omega+\pi))\overline{m_1(\omega)} = 1 \quad \text{for odd } \ell.$$

我們將上述兩方程式改寫成

$$\begin{aligned}\widetilde{m}_0(\omega)\overline{m_0(\omega)} + \widetilde{m}_1(\omega)\overline{m_1(\omega)} &= 1 \\ \widetilde{m}_0(\omega)\overline{m_0(\omega+\pi)} + \widetilde{m}_1(\omega)\overline{m_1(\omega+\pi)} &= 0.\end{aligned}$$

最後我們得到

$$\begin{aligned}\widetilde{m}_0(\omega) &= \frac{\overline{m_1(\omega+\pi)}}{\Delta_{\phi,\psi}(\omega)} \\ \widetilde{m}_1(\omega) &= \frac{-\overline{m_0(\omega+\pi)}}{\Delta_{\phi,\psi}(\omega)},\end{aligned}$$

這裡的 $\Delta_{\phi,\psi}(\omega)$ 定義爲

$$\Delta_{\phi,\psi}(\omega) := \overline{m_0(\omega)m_1(\omega+\pi) - m_0(\omega+\pi)m_1(\omega)}.$$

因爲 $s(\omega)$ 和 $\Gamma(\omega)$ 都屬於 $\mathcal{A}$, 所以 $\Delta_{\phi,\psi}(\omega)$, $\widetilde{m}_0(\omega)$ 與 $\widetilde{m}_1(\omega)$ 也都屬於 $\mathcal{A}$. □

我們可以找到另一個自格函數 $\tilde{\phi}(x)$ 符合 $\widehat{\tilde{\phi}}(\omega) = \widetilde{m}_0(\frac{\omega}{2})\widehat{\tilde{\phi}}(\frac{\omega}{2})$. 亦即, $\tilde{\phi}(x)$ 符合自格等式

$$\tilde{\phi}(x) = \sum \tilde{2}\tilde{c}_n\tilde{\phi}(2x-n). \tag{1.25}$$

稍後我們會討論有關 $(\tilde{c}_n)$ 與 $\tilde{\phi}(x)$ 的一些性質. 在此, 我們令

$$\widetilde{V}_j = \operatorname{span}\{\sqrt{2^j}\tilde{\phi}(x-n)\}.$$

$(\widetilde{V}_j)$ 形成一組 MRA. 現在, 我們令 W_0 是 V_0 在 V_1 下的正交補集, 即

$$\widetilde{V}_1 = \widetilde{V}_0 \oplus \widetilde{W}_0.$$

則存在一 $\tilde{\psi}(x)$ 使得

$$\widetilde{W}_j = \operatorname{span}\{\sqrt{2^j}\tilde{\psi}(x-n)\}.$$

其中 $\widehat{\tilde{\psi}}(\omega) = \widetilde{m}_1(\frac{\omega}{2})\widehat{\tilde{\psi}}(\frac{\omega}{2})$. 如此建造的 $\tilde{\phi}(x)$ 將會是 $\phi(x)$ 的對偶基底 (*dual basis*). 亦即

$$\langle \phi(x), \tilde{\phi}(x-n)\rangle = \delta_{0,n} \quad \forall n \in \mathbb{Z}.$$

上述求數列 (a_n) 與 (b_n) 的方法並不限於半正交凌波函數這一種情形, 事實上, 所有的凌波函數的還原係數都是用如上的方法找的. 當 $\phi(x-n)$ 是正交自格函數時, $\Gamma(\omega) \equiv 1$. 若我們選取 $s(\omega) \equiv 1$, 則

$$\sum |\hat{\psi}(\omega + 2\pi n)|^2 = 1 \quad \forall \omega \in \mathbb{R}.$$

亦即 $\psi(x-n)$ 是 W_0 的一組正交基底. 我們可以發現 $\psi(x)$ 的下降速率是由 $s(\omega)$ 所決定. 若 $s(\omega)$ 的對應係數是有限長的, 則 $\psi(x)$ 是有限涵蓋. 根據 (1.20) 和 (1.22), 我們將 $\widetilde{m}_0(\omega)$ 重寫成 ($s(\omega)$ 被約除了)

$$\widetilde{m}_0(\omega) = \frac{m_1(\omega+\pi)}{\Delta_{\phi,\psi}(\omega)} = m_0(\omega)\frac{\Gamma(\omega)}{\Gamma(2\omega)}. \tag{1.26}$$

就一般而言, $\Gamma(\omega)$ 是不能被 $\Gamma(2\omega)$ 所整除的, 所以 $\tilde{\phi}$ 和 $\tilde{\psi}$ 將不會是有限函蓋. 儘管 $\tilde{\phi}$ 和 $\tilde{\psi}$ 不是有限函蓋, 我們仍有下列性質:

Proposition 1.5. *Let* $\phi(x)$ *be defined as in* (1.1) *and* $\tilde{\phi}$ *be the semi-orthogonornal dual scaling function defined as in* (1.25). *Let* γ_n *be defined by*

$$\gamma_n = \langle \phi(x), \phi(x-n)\rangle \quad \forall n \in \mathbb{Z}.$$

Then, when the support of $\phi(x)$ *is contained in* $[0, N]$, *we have*

$$\phi(x) = \sum_{n=-N+1}^{N-1} \gamma_n \tilde{\phi}(x-n). \tag{1.27}$$

Proof: 根據 (1.26) 和 (1.2), 我們有

$$\Gamma(\omega)\hat{\tilde{\phi}}(\omega) = m_0(\frac{\omega}{2})\Gamma(\frac{\omega}{2})\hat{\tilde{\phi}}(\frac{\omega}{2}).$$

因此

$$\begin{aligned}\Gamma(\omega)\hat{\tilde{\phi}}(\omega) &= \lim_{k\to\infty}\left[\prod_{j=1}^{k} m_0(\frac{\omega}{2^j})\right]\Gamma(\frac{\omega}{2^k})\hat{\tilde{\phi}}(\frac{\omega}{2^k}) \\ &= \hat{\phi}(\omega).\end{aligned}$$

因爲 $\Gamma(\omega) = \sum_{n=-N+1}^{N-1} \gamma_n e^{-i\omega n}$, 我們有

$$\phi(x) = \sum_{n=-N+1}^{N-1} \gamma_n \tilde{\phi}(x-n). \qquad \square$$

上述性質所要說明的是: 儘管 $\tilde{\phi}$ 不是有限函蓋, 有限涵蓋的 $\phi(x)$ 卻可以被有限個 $\tilde{\phi}(x-n)$ 所展開. 所以 $\phi(x-n)$ 的生成集包含於 $\tilde{\phi}(x-n)$ 的生成集. 亦即 $V_0 \subset \tilde{V}_0$.

另一方面, 當 $\Gamma(\omega) \in \mathcal{A}$ 時, $\Gamma^{-1}(\omega)$ 存在且屬於 $\mathcal{A}$. 我們定義一 2π 周期函數 $\tilde{\Gamma}(\omega)$ 之於 $\tilde{\phi}(x)$ 如同 $\Gamma(\omega)$ 之於 $\phi(x)$. 由 (1.26) 我們得到 $\tilde{\Gamma}(\omega) = \Gamma^{-1}(\omega)$. 所以 $\tilde{\Gamma}(\omega)$ 的係數 $(\tilde{\gamma}_n) \in \ell^1$. 當 $\phi(x)$ 與 $\tilde{\phi}(x)$ 的角色互換時, 我們有類似 (1.27) 的性質, 亦即

$$\tilde{\phi}(x) = \sum \tilde{\gamma}_n \phi(x-n). \tag{1.28}$$

由於 $(\tilde{\gamma}_n) \in \ell^1$, $\tilde{V}_0 \subset V_0$, 所以 $V_0 = \tilde{V}_0$. 並且 $\tilde{\phi}(x) \in L^2(\mathbb{R})$

Lemma 1.1. *Let* $\phi(x)$ *be defined as in* (1.1), $\phi(x) \in L^2(R)$ *and* $\tilde{\phi}(x)$ *be the dual of* $\phi(x)$ *in a semi-orthogornal system. If* $(\tilde{\gamma}_n) \in \ell^1$, *then* $\tilde{\phi}(x) \in L^2(\mathbb{R})$.

Proof: 根據 (1.1) 以及 $(\tilde{\gamma}_n) \in \ell^1$,

$$\begin{aligned}\int |\tilde{\phi}(x)|^2\,dx &= \int \tilde{\phi}(x)\overline{\tilde{\phi}(x)}\,dx \\ &= \int\Big(\sum_n \tilde{\gamma}_n\phi(x-n)\Big)\overline{\Big(\sum_m \tilde{\gamma}_m\phi(x-m)\Big)}\,dx \\ &= \sum_{n,m} \tilde{\gamma}_n\tilde{\gamma}_m\langle\phi(x-n),\phi(x-m)\rangle\end{aligned}$$

因爲 $\phi(x)$ 是有限函蓋, 所以只有有限個 (γ_n) 是非零的. 所以我們令 $C = \max\{|\gamma_n|\}$. 因此

$$\begin{aligned}\int |\tilde{\phi}(x)|^2\, dx &\leq C \sum_{n,m} \tilde{\gamma}_n \tilde{\gamma}_m \\ &\leq C(\sum_n |\tilde{\gamma}_n|)(\sum_m |\tilde{\gamma}_m|) < \infty.\end{aligned}$$

□

有關半正交凌波函數的詳細理論, 請參閱崔錦泰與王建中教授的原始論文. [2]

第四節 雙正交凌波函數

我們定義 $\phi(x)$ 和 $\tilde{\phi}(x)$ 如同 (1.1) 以及 (1.25), 而且兩者都是可以展成多重解分析的自格函數. 當我們說 $\{\phi(x), \psi(x), \tilde{\phi}(x), \tilde{\psi}(x)\}$ 爲一組雙正交系統時, 我們指的是:

$$\langle \tilde{\phi}(x), \phi(x-n) \rangle = \langle \tilde{\psi}(x), \psi)x-n) \rangle = \delta_{0,n} \tag{1.29}$$

以及

$$\langle \tilde{\phi}(x), \psi(x) \rangle = \langle \tilde{\psi}(x), \phi(x) \rangle = 0. \tag{1.30}$$

因此, 對於任意的整數 n 我們有

$$\begin{aligned}\delta_{0,n} &= \langle \tilde{\phi}(x), \phi(x-n) \rangle \\ &= \int \hat{\tilde{\phi}}(\omega)\overline{\hat{\phi}(\omega)} e^{i\omega n}\, d\omega \\ &= \int_0^{2\pi} e^{i\omega n} \sum_n \hat{\tilde{\phi}}(\omega + 2\pi n)\overline{\hat{\phi}(\omega + 2\pi n)}\, d\omega.\end{aligned}$$

由於 n 是任意的, 所以

$$\sum \hat{\tilde{\phi}}(\omega + 2\pi n)\overline{\hat{\phi}(\omega + 2\pi n)} = (2\pi)^{-1} \quad \text{a.e.} \qquad (1.31).$$

我們將 (1.2) 代入上式 (令 $m_0 = m_0^{\phi}$, $\widetilde{m}_0 = m_0^{\tilde{\phi}}$) 得到

$$\sum_n \widetilde{m}_0(\omega + \pi n)\overline{m_0(\omega + \pi n)}\tilde{\phi}(\omega + \pi n)\overline{\phi(\omega + \pi n)} = (2\pi)^{-1} \quad \text{a.e..}$$

同樣的, 我們再將奇數項與偶數項拆開並配合 (1.31), 我們有

$$\widetilde{m}_0(\omega)\overline{m_0(\omega)} + \widetilde{m}_0(\omega + \pi)\overline{m_0(\omega + \pi)} = 1 \quad \text{a.e..} \tag{1.32}$$

仿造上述方法, 根據 (1.29) 和 (1.30) 我們得到

$$
\begin{aligned}
\tilde{m}_0(\omega)\overline{m_1(\omega)} + \tilde{m}_0(\omega+\pi)\overline{m_1(\omega+\pi)} &= 0 \\
\tilde{m}_1(\omega)\overline{m_0(\omega)} + \tilde{m}_1(\omega+\pi)\overline{m_0(\omega+\pi)} &= 0 \\
\tilde{m}_1(\omega)\overline{m_1(\omega)} + \tilde{m}_1(\omega+\pi)\overline{m_1(\omega+\pi)} &= 1 \quad \text{a.e.}.
\end{aligned}
$$

這四條式子是雙正交凌波函數最重要的式子. 只要有兩套自格與凌波函數的特徵函數 $m_0, m_1, \tilde{m}_0, \tilde{m}_1$ 滿足上述四個式子, 則此兩套自格凌波函數自動形成一組對偶的雙正交基底. 綜合以上結果, 我們定義

$$
M(\omega) = \begin{bmatrix} m_0(\omega) & m_0(\omega+\pi) \\ m_1(\omega) & m_1(\omega+\pi) \end{bmatrix}, \quad \widetilde{M}(\omega) = \begin{bmatrix} \tilde{m}_0(\omega) & \tilde{m}_0(\omega+\pi) \\ \tilde{m}_1(\omega) & \tilde{m}_1(\omega+\pi) \end{bmatrix}.
$$

若

$$
\widetilde{M}(\omega)\overline{M^t(\omega)} = 1 \quad \text{a.e.}, \tag{1.33}
$$

則對應的兩套自格凌波函數自動形成一組對偶的雙正交基底.

同樣的, 在雙正交凌波函數的情況下, 凌波函數也不是被唯一決定的. 我們有下列定理供讀者參考:

Theorem 1.3. *Let* ϕ *and* $\tilde{\phi}$ *be a pair of biorthogonal scaling functions, and* $\{m_0, \tilde{m}_0\}$ *be the corresponding* 2π*-periodic characteristic functions. Then* $\{m_1, \tilde{m}_1\}$ *satisfy* (1.33) *if and only if they are chosen from the class*

$$
m_1(\omega) = e^{-i\omega}\overline{\tilde{m}_0(\omega+\pi)}s^{-1}(2\omega) \quad \text{and} \quad \tilde{m}_1(\omega) = e^{-i\omega}\overline{m_0(\omega+\pi)}s(2\omega),
$$

for some $s \in \mathcal{A}$.

有關雙正交凌波理論的詳細內容, 請參閱 Cohen, Daubechies, 和 Feauveau 的原始論文 [3].

An open problem. 回顧前面介紹的三種形態的凌波函數, 我們知道正交凌波函數可視爲半正交凌波函數的一個特例; 而半正交凌波函數可視爲雙正交凌波函數的一個特例. 在考慮雙正交系統時 $m_0(\omega)$ 與 $\widetilde{m_0}(\omega)$ 必滿足 (1.32), 且

$$
m_1(\omega) = e^{-i\omega}\overline{\widetilde{m_0}(\omega+\pi)}s(2\omega). \tag{1.34}
$$

我們將 (1.34) 代入 (1.19) 得到

$$\Big(m_0(\omega)\widetilde{m_0}(\omega+pi)\Gamma(\omega)-m_0(\omega+pi)\widetilde{m_0}(\omega)\Gamma(\omega+\pi)\Big)e^{i\omega}\overline{s(2\omega)}=0.$$

所以我們考慮此聯立方程式:

$$\begin{cases}\tilde{m}_0(\omega)\overline{m_0(\omega)}+\widetilde{m_0}(\omega+\pi)\overline{m_0(\omega+\pi)}=1\\ m_0(\omega)\widetilde{m_0}(\omega+\pi)\Gamma(\omega)-m_0(\omega+\pi)\widetilde{m_0}(\omega)\Gamma(\omega+\pi)=0\end{cases}$$

當 $m_0(\omega)$ 爲某些 B-spline 的特徵函數時, 我們曾嘗試去求有限長係數的 $\widetilde{m_0}(\omega)$. 不幸地, 我們並沒有成功. 在現有的半正交系統中, 我們只考慮 $\phi(x)$ 是 B-spline 的情形. 而其對應的對偶自格函數 $\tilde{\phi}(x)$ 的函蓋是無窮長的. 我們相信當我們對 $\phi(x)$ 以及 $\tilde{\phi}(x)$ 放開一些條件時, 如對稱性或逼近能力, 則我們可以求出有限函蓋的半正交凌波系統. 如此, 整個凌波理論會顯得更統一更完整.

第五節 拉舉法

我們令 $\{\phi,\psi,\tilde{\phi},\tilde{\psi}\}$ 構成一組雙正交系統. 即

$$\langle\phi(x),\tilde{\phi}(x-n)\rangle=\delta_{0,n}\ \text{以及}\ \langle\psi,\tilde{\psi}(x-n)\rangle=\delta_{0,n},$$

且

$$\langle\phi(x),\tilde{\psi}(x-n)\rangle=0\ \text{以及}\ \langle\tilde{\phi}(x),\psi(x-n)\rangle=0.$$

在雙正交系統中必有 (1.32), 現在我們將它寫成以下這種型式:

$$m_0^{\tilde{\phi}}(\omega)\overline{m_0^{\phi}(\omega)}+m_0^{\tilde{\phi}}(\omega+\pi)\overline{m_0^{\phi}(\omega+\pi)}=1\quad \text{a.e..} \tag{1.35}$$

若我們新定義一自格函數 $\tilde{\phi}_1(x)$, 其特徵多項式爲

$$m_0^{\tilde{\phi}_1}(\omega)=m_0^{\tilde{\phi}}(\omega)+e^{-i\omega}\overline{m_0^{\phi}(\omega+\pi)s(2\omega)},$$

這裡的 $s(\omega)$ 是一個屬於 $\mathcal{A}$ 的三角多項式. 則 $\tilde{\phi}_1$ 也會是一個對應於 $\phi(x)$ 的對偶自格函數. 我們可以很容易的檢驗 $m_0^{\tilde{\phi}_1}(\omega)$ 也滿足 (1.35), 即

$$\langle\phi(x),\tilde{\phi}_1(x-n)\rangle=\delta_{0,n}.$$

當 $\phi(x)$ 與 $\tilde{\phi}(x)$ 被決定後, 我們導出新的凌波函數.

由 Theorem 1.3, m_1^{ϕ} 是被一個屬於 $\mathcal{A}$ 的 $s(2\omega)$ 所決定的, 且 Wiener class 是一個代數結構, 所以 $m_0^{\tilde{\phi}_1}$ 可以寫成

$$m_0^{\tilde{\phi}_1}(\omega) = m_0^{\tilde{\phi}}(\omega) + m_1^{\tilde{\phi}}(\omega)\overline{s(2\omega)},$$

且

$$m_1^{\phi_1}(\omega) = m_1^{\phi}(\omega) - m_0^{\phi}(\omega)s(2\omega).$$

由此方法我們可以把原來的特徵函數 $m_0^{\phi}(\omega)$, $m_1^{\phi}(\omega)$, $m_0^{\tilde{\phi}}(\omega)$ 和 $m_1^{\tilde{\phi}}(\omega)$ 變成新的 $m_0^{\phi}(\omega)$, $m_1^{\phi_1}(\omega)$, $m_0^{\tilde{\phi}_1}(\omega)$ 和 $m_1^{\tilde{\phi}}(\omega)$. 舊有的雙正交系統 $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ 變爲新的雙正交系統 $\{\phi, \psi_1, \tilde{\phi}_1, \tilde{\psi}_1\}$. 我們注意到: (1) 因爲 ψ_1 仍舊在 V_1 空間裡, 所以 ψ_1 的平滑度並沒有改變; (2) $\tilde{\psi}_1$ 的凌波係數與 $\tilde{\psi}$ 的相同, 它們都是由 (c_n) 所決定的. 要注意的是, 雖然 $\tilde{\psi}_1$ 的凌波係數與 $\tilde{\psi}$ 的相同, 但 $\tilde{\phi}_1$ 已經改變了, 所以 $\tilde{\psi}_1$ 與 $\tilde{\psi}$ 本質上已經不同了.

經由適當的選取 $s(\omega)$, 我們可以增加 ψ_1 的消散數 (vanishing moments), 亦即 $\psi(x)$ 正交於多項式的階數. 且 $\tilde{\phi}_1$, $\tilde{\psi}_1$ 可以有較高的平滑度, 亦即 $\tilde{\phi}_1$, $\tilde{\psi}_1$ 有較高的可微程度. 這就是所謂對於雙正交系統的拉舉過程 (*lifting scheme*). 總結上述, 我們有下列定理:

Theorem 1.4. (Lifting Scheme). *Let* $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ *be an initial set of biorthogonal system. Then a new set* $\{\phi, \tilde{\phi}_1, \psi_1, \tilde{\psi}_1\}$, *which is formally biorthogonal, can be constructed by*

$$\begin{aligned}
\psi_1(x) &= \psi(x) - \sum s_n \phi(x-n) \\
\tilde{\phi}_1(x) &= \sum \tilde{c}_n \tilde{\phi}_1(2x-n) + \sum s_{-n} \tilde{\psi}_1(x-n) \\
\tilde{\psi}_1(x) &= \sum d_n \tilde{\phi}_1(2x-n).
\end{aligned}$$

這裡的 $\tilde{c}_n$ 是 $\tilde{\phi}$ 的自格係數, d_n 是 $\tilde{\psi}$ 的凌波係數以及 s_n 是三角多項式 $s(\omega)$ 的係數.

有關拉舉法的詳細內容, 請參閱 Sweldens 的原始論文 [8].

第二章

一套提昇自格函數與凌波函數逼近能力與平滑度的方法

在這一章我們要介紹一種簡單的提昇凌波函數逼近能力與平滑度的方法. 首先, 在沒有特別聲明的情況下, 我們的自格函數 $\phi(x)$ 是指 Daubechies 的正交自格函數. 在 Daubechies 的正交自格函數裡, 我們令 $\phi(x)$ 滿足自格等式 (1.1), $c_0, c_1, \ldots, c_{2p-1}$ 是相對應於 $\phi(x)$ 的自格係數. 常數 p 是凌波函數的貼近度, 亦即 $\phi(x-n)$ 的生成集包含了所有 p 階以下的多項式. 在這裡我們指的是踽限在某一個區間上 $\phi(x-n)$ 的生成集可以將踽限在這一個區間上的 p 階以下的多項式展開. 我們令 $\psi(x)$ 是對應於 $\phi(x)$ 的正交凌波函數, 則

$$\langle x^m, \psi(x)\rangle = 0, \quad 0 \le m \le p-1.$$

當 $\phi(x)$ 是 Daubechies 的正交自格函數時, 自格係數 c_n 滿足下列等式:

$$\begin{aligned}
&(S)\ \sum_{n=0}^{2p-1} c_n = 2,\\
&(A)\ \sum_{n=0}^{2p-1} (-1)^n n^m c_n = 0 \text{ where } m = 0, \ldots, p-1 \quad (0^0 := 1),\\
&(O)\ \sum_{n=0}^{2p-1} c_n c_{n-2m} = \delta_{0m}, \text{ for all } m \in \mathbb{Z}.
\end{aligned} \tag{2.1}$$

此時, $\phi(x)$ 的涵蓋是 $[0, 2p-1]$. 當我們令 (d_n) 是 $\psi(x)$ 的凌波係數, 則 $d_n = (-1)^n c_{1-n}$, 所以 $\operatorname{supp}\psi(x) = [1-p, p]$.

我們依從 [10] 那篇文章裡的想法, 定義

$$\Phi(x)=\int_{-\infty}^{x}\phi(t)-\phi(t-1)\,dt. \tag{2.2}$$

我們稱 $\Phi(x)$ 是 $\phi(x)$ 的提昇. 我們將要證明 $\Phi(x)$ 也是自格函數, 且 $\Phi(x)$ 也可以展出一多重解分析. 當然, 我們可以由此多重解分析導出對應於 $\Phi(x)$ 的凌波函數 $\Psi(x)$. 提昇後的 $\Phi(x)$ 與 $\Psi(x)$ 都不再具有正交性. 但是, 當我們選取最簡單的造凌波函數的方法, 即

$$g_n=(-1)^n h_{1-n},$$

其中 (h_n) 是 $\Phi(x)$ 的自格係數, (g_n) 是 $\Psi(x)$ 的凌波係數. 我們得到 $\Psi(x-n)$ 是可微的, 並且 $\Psi'(x-n)$ 是一組正交的函數. 因爲提昇後的 $\Phi(x)$ 與 $\Psi(x)$ 都不再具有正交性, 所以我們只考慮雙正交系統. 我們將造出對偶的 $\{\tilde{\Phi}(x),\tilde{\Psi}(x)\}$, 使得我們有快速的演算法在 V_{j+1} 與 V_j+W_j 間作基底變換.

第一節 $\Phi(x)$ 是自格函數

由 $\Phi(x)$ 的定義得知

$$\begin{aligned}\Phi(x)&=\int_{-\infty}^{x}\phi(t)-\phi(t-1)\,dt\\&=\int_{-\infty}^{x}[\sum_{n=0}^{2p-1}c_n\phi(2t-n)]-[\sum_{n=0}^{2p-1}c_n\phi(2t-n-2)]\,dt\\&=\int_{-\infty}^{x}\sum_{n=0}^{2p}\frac{(c_{n-1}+c_n)}{2}[\phi(2t-n-2)-\phi(2t-n-1)]\,dt\\&:=\sum_{n=0}^{2p}h_n\Phi(2x-n).\end{aligned}$$

我們可以看到提昇後的 $\Phi(x)$ 的自格係數 (h_n) 爲

$$h_n=\frac{(c_{n-1}+c_n)}{2}, \tag{2.3}$$

注意 $c_{-1}=c_{2p}=0$. $\Phi(x)$ 滿足自格等式且非零的自格係數有 $2p+1$ 個, 所以 $\operatorname{supp}\Phi(x)=[0,2p]$. 再來, 我們可以很容易地看見以下的關係

$$\sum_{n=0}^{2p}h_n=\sum_{n=0}^{2p-1}c_n=2, \tag{2.4}$$

和

$$\sum_{n=0}^{2p}(-1)^n h_n = 0.$$

Proposition 2.1. *If* $\phi(x)$ *is a scaling function that satisfies* (1.1) *and* $\Phi(x)$ *is defined as in* (2.2)*. Then*

$$\Phi(x) = \chi_{[0,1]} * \phi(x). \tag{2.5}$$

Furthermore, $\Phi(x)$ *is a scaling function with scaling coefficients*

$$(h_n) = (\frac{1}{2}, \frac{1}{2}) * (c_n).$$

Proof: 由 (2.2) 以及疊積的性質,

$$\begin{aligned}
\Phi(x) &= \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt \\
&= \int_{-\infty}^{x} \phi(t)\, dt - \int_{-\infty}^{x} \phi(t-1)\, dt \\
&= \int_{-\infty}^{x} \phi(t)\, dt - \int_{-\infty}^{x-1} \phi(t)\, dt \\
&= \int_{x-1}^{x} \phi(t)\, dt \\
&= \int_{-\infty}^{\infty} \chi_{[0,1]}(x-t)\phi(t)\, dt \\
&= \chi_{[0,1]} * \phi(x).
\end{aligned}$$

因此我們有

$$\begin{aligned}
\hat{\Phi}(\omega) &= \widehat{\chi_{[0,1]}}(\omega) \cdot \hat{\phi}(\omega) \\
&= \frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} e^{-i\frac{\omega}{2}}\, \hat{\phi}(\omega) \\
&= (\frac{1}{2} + \frac{1}{2}e^{-i\frac{\omega}{2}})\widehat{\chi_{[0,1]}}(\frac{\omega}{2})\, m_0^{\phi}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}) \\
&:= m_0^{\Phi}(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2}).
\end{aligned}$$

所以

$$\begin{aligned} m_0^{\Phi}(\omega) &= (\frac{1}{2}+\frac{1}{2}e^{-i\omega})m_0^{\phi}(\omega) \\ &= \frac{1}{2}(\frac{1}{2}+\frac{1}{2}e^{-i\omega})\sum_n c_n e^{-in\omega} \\ &:= \frac{1}{2}\sum_n h_n e^{-in\omega}. \end{aligned}$$

從這裡我們注意到 (h_n) 是由 $(\frac{1}{2},\frac{1}{2})$ 與 (c_n) 作離散型的疊積而來的. 我們將

$$\hat{\Phi}(\omega) = m_0^{\Phi}(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2}),$$

作逆傅氏變換得到

$$\Phi(x) = \sum_n h_n \Phi(2x-n).$$

所以 $\Phi(x)$ 也是自格函數. □

Remark. 我們知道 $\widehat{\chi_{[0,1]}}(\omega)$ 是在 0 點連續. 因爲 $\hat{\phi}(\omega)$ 也是連續函數, 所以 $\Phi(\omega)$ 也是連續函數. 因此我們有 $\hat{\Phi}(0)=1$ 以及, 對於所有整數 $n \neq 0$, $\hat{\Phi}(2n\pi)=0$.

第二節 $\Phi(x)$ 展出一多重解分析

從 (c_n) 滿足 (2.1) 的三個式子來看, (S) 是由 (1.1) 所推得的, (A) 是有 p 階逼近度的充份必要條件, (O) 是自格函數滿足正交性的必要條件. 在 (2.4), 我們已看見 (h_n) 滿足 (S). 我們將要證明 (h_n) 滿足更高一階的 (A), 但 (h_n) 不會滿足 (O).

Proposition 2.2. *If* $\phi(x)$ *is a scaling function that satisfies* (1.1) *and* (c_n) *satisfies the condition* (A) *in* (2.1), *and if* $\Phi(x)$ *is as defined in* (2.2). *Than the scaling coefficients* (h_n) *of* $\Phi(x)$ *satisfy*

$$\sum_{n=0}^{2p}(-1)^n n^m h_n = 0, \text{ where } m = 0,\dots,p \text{ (with } 0^0 := 1). \tag{2.6}$$

Proof: 對於 $m = 0$,

$$\begin{aligned}
\sum_{n=0}^{2p}(-1)^n h_n &= \sum_{n=0}^{2p}(-1)^n(\frac{c_n + c_{n-1}}{2}) \\
&= \frac{1}{2}[\sum_{n=0}^{2p-1}(-1)^n c_n + \sum_{n=0}^{2p-1}(-1)^{n+1}c_n] \\
&= \frac{1}{2}[\sum_{n=0}^{2p-1}(-1)^n c_n - \sum_{n=0}^{2p-1}(-1)^n c_n] \\
&= 0.
\end{aligned}$$

假設對於 $m = 0, \ldots, k-1$ 式 (2.6) 都滿足, 其中 $0 < k \le p$. 則

$$\begin{aligned}
\sum_{n=0}^{2p}(-1)^n n^k h_n &= \sum_{n=0}^{2p}(-1)^n n^k(\frac{c_n + c_{n-1}}{2}) \\
&= \frac{1}{2}\sum_{n=0}^{2p}(-1)^n n^k c_n + \frac{1}{2}\sum_{n=0}^{2p}(-1)^n n^k c_{n-1} \\
&= \frac{1}{2}\sum_{n=0}^{2p-1}(-1)^n n^k c_n + \frac{1}{2}\sum_{n=1}^{2p}(-1)^n n^k c_{n-1} \\
&= \frac{1}{2}[\sum_{n=0}^{2p-1}(-1)^n n^k c_n + \sum_{n=0}^{2p-1}(-1)^{n+1}(n+1)^k c_n] \\
&= \frac{1}{2}\sum_{n=0}^{2p-1}(-1)^n[n^k - (n+1)^k]c_n \\
&= \frac{1}{2}\sum_{n=0}^{2p-1}(-1)^{n+1}[\sum_{\ell=0}^{k-1}\binom{k}{\ell}n^\ell]c_n \\
&= -\frac{1}{2}\sum_{\ell=0}^{k-1}\binom{k}{\ell}[\sum_{n=0}^{2p-1}(-1)^n n^\ell c_n] \\
&= 0.
\end{aligned}$$

因此, 由數學歸納法我們證明了 (2.6). □

由上述性質得知, $\Phi(x-n)$ 的生成集包含了 p 階以及 p 階以下的多項式. 由此可見, $\Phi(x)$ 的逼近能力增加了. 因爲 $\Phi(x)$ 的自格係數滿足

$$\sum_n h_n = 2,$$

所以我們有下列性質:

Proposition 2.3. *If* $\phi(x)$ *is a scaling function such that*

$$\sum_{n\in\mathbb{Z}} \phi(x-n) = 1, \quad \forall x \in \mathbb{R},$$

and $\Phi(x)$ *is as defined in* (2.2). *Then*

$$\sum_{n\in\mathbb{Z}} \Phi(x-n) = 1, \quad \forall x \in \mathbb{R}.$$

除此之外

$$\sum_{n\in\mathbb{Z}} n^m \Phi(x-n)$$

會是一個 m 階的多項式, $0 \le m \le p \quad (0^0 := 1)$.

我們令 $V_0 = \text{span}\,\{\Phi(x-n)\}$, 我們要證明是否 $\Phi(x-n)$ 是 V_0 的一組 Riesz basis. 要證明此性質我們只要證明: 存在常數 $0 < A \le B$ 使得

$$A \le \sum_{n=-\infty}^{\infty} |\hat{\Phi}(\omega + 2\pi n)|^2 \le B, \quad \forall \omega \in \mathbb{R}.$$

若 $\Phi(x-n)$ 是 V_0 的一組 Riesz basis 且 $\hat{\Phi}(0) \neq 0$, 則 $\Phi(x)$ 可以展出一多重解分析.

Theorem 2.1. *If* ϕ *is a scaling function that generates a multiresolution anaysis and* $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *then the function* $\Phi = \phi * \chi_{[0,1]}$ *also generates a multiresolution analysis.*

在證明此一性質前, 我們先證明下面的定理與引理.

Theorem 2.2. *Let* $f \in L^1(\mathbb{R})$. *Then its Fourier transform* $\hat{f}$ *satisfies:*

(1) $\hat{f} \in L^\infty(\mathbb{R})$ *with* $\|\hat{f}\|_\infty \le \|f\|_1$;

(2) $\hat{f}$ *is uniformly continuous on* $\mathbb{R}$;

(3) *if the derivative* f' *of* f *also exists and is in* $L^1(\mathbb{R})$, *then*

$$\widehat{f'}(\omega) = i\omega\hat{f}(\omega); \text{ and } \hat{f}(\omega) \to 0, \text{ as } |\omega| \to \infty.$$

Proof: (1) 式是簡單的. 對於任意的 $\omega \in \mathbb{R}$

$$\begin{aligned}|\hat{f}(\omega)| = |\int e^{-i\omega x} f(x)\,dx| &\le \int |e^{-i\omega x} f(x)|\,dx \\ &\le \|e^{-i\omega x}\|_\infty \|f\|_1 \le \|f\|_1.\end{aligned}$$

證明 (2), 令 δ 是任意選取的, 且考慮

$$\begin{aligned}\sup_{\omega}|\hat{f}(\omega+\delta)-\hat{f}(\omega)| &= \sup_{\omega}\left|\int e^{-i\omega x}(e^{-i\delta x}-1)f(x)\,dx\right| \\ &\leq \int |e^{-i\delta x}-1|\,|f(x)|\,dx.\end{aligned}$$

因爲 $|e^{-i\delta x}-1|\,|f(x)| \leq 2|f(x)| \in L^1(\mathbb{R})$ 以及 $|e^{-i\delta x}-1| \to 0$ 當 $\delta \to 0$. 由 Lebesgue Dominated Convergence Theorem 得知, $\sup_{\omega}|\hat{f}(\omega+\delta)-\hat{f}(\omega)| \to 0$ 當 $\delta \to 0$. 所以 $\hat{f}$ 在 $\mathbb{R}$ 上是均勻連續的.

證明 (3), 我們利用標準的分部積分法:

$$\begin{aligned}\widehat{f'}(\omega) &= \int e^{-i\omega x}f'(x)\,dx = \int e^{-i\omega x}\,df(x) \\ &= e^{-i\omega x}f(x)\Big|_{-\infty}^{\infty} - \int f(x)\,de^{-i\omega x} = i\omega\int e^{-i\omega x}f(x)\,dx \\ &= i\omega\hat{f}(\omega),\end{aligned}$$

因爲 $f \in L^1(\mathbb{R})$, $f(x) \to 0$ 當 $|x| \to \infty$. 若 f' 屬於 $L^1(\mathbb{R})$, 利用式 (1)

$$|\hat{f}(\omega)| = \frac{1}{|\omega|}|\widehat{f'}(\omega)| \leq \frac{1}{|\omega|}\|f'\|_1 \to 0,$$

當 $|\omega| \to \infty$. □

Lemma 2.1. *Let* $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and* $\Phi = \chi_{[0,1]} * \phi$. *Then the function*

$$\Gamma(\omega) := \sum_{n=-\infty}^{\infty} |\hat{\Phi}(\omega+2\pi n)|^2, \quad \textit{for all } \omega \in \mathbb{R} \tag{2.7}$$

is a continuous function of ω *on* $(-2\pi, 2\pi)$.

Proof: 因爲 $\Phi(x) \in L^2(R)$, 所以 $\Gamma(\omega)$ 是一個全函數, 所以 $\Gamma(\omega)$ 是一個連續函數. □

Lemma 2.2. *If the function* $\Gamma(\omega)$ *is defined as* (2.7), *then* $\Gamma(\omega) > 0$ *for all* $\omega \in [0, 2\pi)$.

Proof: 若存在 $\omega_0 \in [0, 2\pi)$ 使得 $\Gamma(\omega_0) = 0$, 則

$$\sum_{n=-\infty}^{\infty}\left(\frac{\sin\frac{\omega_0}{2}}{\frac{\omega_0+2\pi n}{2}}\right)^2 |\hat{\phi}(\omega_0+2\pi n)|^2 = 0.$$

我們有 $\sin\frac{\omega_0}{2}=0$ 或 $\hat{\phi}(\omega_0+2\pi n)=0$, 對於所有的 $n\in\mathbb{Z}$. 若 $\sin\frac{\omega_0}{2}=0$, 則 $\omega_0=0$. 根據，得到

$$\hat{\Phi}(0)=1 \text{ and } \hat{\Phi}(2\pi n)=0 \quad \text{for } n\neq 0,$$

所以 $\Gamma(0)=1$. 那麼 $\Gamma(\omega_0)=0$ 就只有可能是 $\hat{\phi}(\omega_0+2\pi n)=0 \quad \forall n\in\mathbb{Z}$. 這與

$$0<A_\phi\le\sum_{n=-\infty}^{\infty}|\hat{\phi}(\omega_0+2\pi n)|^2$$

矛盾. 所以 $\Gamma(\omega)>0$, $\forall\omega\in[0,2\pi)$. □

現在我們可以開始證明 Theorem 2.1

Proof: 不等式 $\sum_{n=-\infty}^{\infty}|\hat{\Phi}(\omega+2\pi n)|^2\le B$ 是簡單的. 因爲 $\phi(x-n)$ 是 Riesz basis, 因此存在常數 $0<A_\phi\le B_\phi$ 使得

$$A_\phi\le\sum_{n=-\infty}^{\infty}|\hat{\phi}(\omega+2\pi n)|^2\le B_\phi.$$

由於

$$|\hat{\Phi}(\omega+2\pi n)|^2=\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega+2\pi n}{2}}\right)^2|\hat{\phi}(\omega+2\pi n)|^2\le|\hat{\phi}(\omega+2\pi n)|^2$$

對於所有的 $\omega\in\mathbb{R}$ 以及 $n\in\mathbb{Z}$. 我們有

$$\sum_{n=-\infty}^{\infty}|\hat{\Phi}(\omega+2\pi n)|^2\le\sum_{n=-\infty}^{\infty}|\hat{\phi}(\omega+2\pi n)|^2\le B_\phi.$$

在另一方面, 因爲 Γ 是一個 2π 週期的連續函數, 且 Γ 的函數值恆大於零, 故 Γ 在 $[0,2\pi]$ 區間上有一個大於零的最小值. 因此存在一常數 $A_\Phi>0$ 使得 $A_\Phi\le\Gamma(\omega) \quad \forall\omega\in\mathbb{R}$. 我們證明了 $\Phi(x-n)$ 是一個 Riesz basis. □

第三節 $\Psi'(x-n)$ 是正交集合

由上節我們知道 $\Phi(x)$ 也展成一多重解分析. 此時此刻我們尚不知道 $\Phi(x-n)$ 是否是一個正交集合. 在這一節後半段我們將要證明用此種提昇法, 不可能將一個正交的自格函數提昇爲另一個正交的自格函數. 因此, 我們只考慮一般的雙正交系統, 請參考 (1.32). 我們將從 $\Phi(x)$ 的對偶自格函數 $\tilde{\Phi}(x)$ 來導出對應於 $\Phi(x)$ 的凌波函數. 新的雙正交系統 $\{\Phi,\Psi,\tilde{\Phi},\tilde{\Psi}\}$ 也必須滿足 (1.33). 因爲

$$m_0^\Phi(\omega)=\left(\frac{1+e^{-i\omega}}{2}\right)m_0^\phi(\omega), \tag{2.8}$$

我們選擇

$$m_0^{\tilde{\Phi}}(\omega) = \begin{cases} \dfrac{2e^{-i\omega}}{1+e^{-i\omega}} m_0^{\phi}(\omega) & \text{if } \omega \neq (2n+1)\pi, \\ 0 & \text{otherwise,} \end{cases}$$

以及

$$m_1^{\tilde{\Phi}}(\omega) = e^{-i\frac{\omega}{2}}\overline{m_0^{\Phi}(\omega+\pi)}, \qquad m_1^{\Phi}(\omega) = e^{-i\frac{\omega}{2}}\overline{m_0^{\tilde{\Phi}}(\omega+\pi)}.$$

如此我們便造出新的雙正交系統 $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$. 從這一個特殊的選擇下, 我們將要看到 (1) $\tilde{\Phi}$ 之於 $\tilde{\phi}$ 的關係與 Φ 之於 ϕ 的關係是交互的; (2) $\Psi'(x-n)$ 是一個正交集合且正交於 $\Phi'(x-n)$; (3) $\Phi(x-n)$ 不是正交集合.

Proposition 2.4. *Let ϕ be an orthonormal scaling function and let Φ be defined by* (2.5)*. If $\tilde{\Phi}$ is defined by*

$$\chi_{[-1,0]} * \tilde{\Phi} = \phi. \tag{2.9}$$

Then $\tilde{\Phi}$ is a biorthogonal dual of Φ.

Proof: 若存在一函數 $\tilde{\Phi}(x)$ 使得 $\chi_{[-1,0]} * \tilde{\Phi} = \phi$, 則

$$\widehat{\chi_{[-1,0]}}(\omega)\hat{\tilde{\Phi}}(\omega) = \hat{\phi}(\omega),$$

亦即

$$\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)e^{i\frac{\omega}{2}}\hat{\tilde{\Phi}}(\omega) = \hat{\phi}(\omega).$$

因此

$$m_0^{\tilde{\Phi}}(\omega) = \frac{2}{1+e^{i\omega}} m_0^{\phi}(\omega) \quad \text{a.e.}. \tag{2.10}$$

理所當然 $m_0^{\Phi}(\omega)$ 與 $m_0^{\tilde{\Phi}}(\omega)$ 滿足 (1.32), 且 $\tilde{\Phi}(x)$ 對偶於 $\Phi(x)$. $\square$

Remark. 在實際應用上我們所遇見的 $m_0^{\tilde{\Phi}}(\omega)$ 大多有 $(1+e^{-i\omega})/2$ 因子. 所以 $m_0^{\tilde{\Phi}}(\omega)$ 在實際應用上只有有限個係數; 換句話說 $\tilde{\Phi}(x)$ 是有限涵蓋的. 然而, 從 (2.9) 我們看到 $\tilde{\Phi}(x)$ 的平滑度會比 $\tilde{\phi}(x)$ 還來的低. 我們在 [3] 這一篇論文裡也看到同樣的現象.

在應用上我們並不太在意 $\{\tilde{\Phi}, \tilde{\Psi}\}$ 是否屬於 $L^2(\mathbb{R})$. 只要對偶的自格係數與凌波係數是有限長的. 有限長的係數才能幫助我們有快速的基底變換. 無論 $\tilde{\Phi}$ 與 $\tilde{\Psi}$ 是否屬於 $L^2(\mathbb{R})$, 至少它們是個分怖 (distribution).

我們知道 $\Phi(x)=\int_{-\infty}^{x}\phi(t)-\phi(t-1)\,dt$ 若且唯若 $\Phi=\phi*\chi_{[0,1]}$. 我們可以簡單的看到 $\chi_{[-1,0]}*\tilde{\Phi}=\tilde{\phi}$ 若且爲若 $\tilde{\phi}'(x)=\tilde{\Phi}(x+1)-\tilde{\Phi}(x)$. 當 $\tilde{\Phi}$ 是有限涵蓋且我們知道 $\tilde{\phi}'(x)$ 的函數值時, 我們可以很容易的把 $\tilde{\Phi}(x)$ 的函數值算出來.

下列性質將展式 Ψ 與原始的凌波函數 ψ 間美妙的關係, 亦即

$$\Psi(x)=4\int_{-\infty}^{x}\psi(t)\,dt \quad \forall x\in\mathbb{R}. \tag{2.11}$$

Proposition 2.5. *Let* $\Phi(x)$ *be the elevation of the orthonormal scaling function* $\phi(x)$ *and* $\tilde{\Phi}(x)$ *is defined by* (2.9)*. If we construct the wavelet* $\Psi(x)$ *from* $m_1^{\Phi}(\omega)$*, then* $\Psi'(x-n)$ *is an orthogonal set.*

Proof: 因爲 $\Phi(x)=\chi_{[0,1]}*\phi(x)$ 且 $\phi(x)=\chi_{[-1,0]}*\tilde{\Phi}(x)$, 我們有 $m_0^{\Phi}(\omega)$ 滿足 (2.8) 及 $m_0^{\tilde{\Phi}}(\omega)$ 滿足 (2.10). 因此

$$\begin{aligned} m_1^{\Phi}(\omega) &= e^{-i\frac{\omega}{2}}\left(\frac{1+e^{-i(\omega+\pi)}}{2}\right)^{-1}\overline{m_0(\omega+\pi)} \\ &= m_1(\omega)\left(\frac{1+e^{-i(\omega+\pi)}}{2}\right)^{-1}. \end{aligned}$$

由 Theorem 2.2, $\widehat{\Psi'}(\omega)=i\omega\hat{\Psi}(\omega)$, 所以

$$\begin{aligned} i\omega\hat{\Psi}(\omega) &= i\omega\, m_1(\frac{\omega}{2})\left(\frac{1+e^{-i(\frac{\omega}{2}+\pi)}}{2}\right)^{-1}\hat{\Phi}(\frac{\omega}{2}) \\ &= i\omega\, m_1(\frac{\omega}{2})\left(\frac{2}{1-e^{-i\frac{\omega}{2}}}\right)\frac{2(1-e^{-i\frac{\omega}{2}})}{i\omega}\hat{\phi}(\frac{\omega}{2}) \\ &= 4\, m_1(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}) \\ &= 4\,\hat{\psi}(\omega) \quad \text{a.e..} \end{aligned}$$

將上述等式兩邊作逆傅氏變換得到

$$\Psi'(x)=4\psi(x), \quad \forall x\in\mathbb{R}.$$

□

Remark. 令 $\phi(x)$ 是 p 階的 Daubechies 的正交自格函數. 對於 $1\le r<p$, 我們定義 $\phi_r(x)$ 爲

$$\phi_r(x)=\phi*\chi_{[0,1]}*\ldots*\chi_{[0,1]} \quad (\text{r times}),$$

若 $\{\phi_r, \psi_r, \tilde{\phi}_r, \tilde{\psi}_r\}$ 是一組雙正交系統, 其導出的方法如同前面所提的方式, 則

$$\psi_r^{(r)}(x) = 4^r \psi(x).$$

亦即, $\psi_r^{(r)}(x-n)$ 是正交集合.

現在, 我們要證明從上述方法中所造出來的 $\Phi(x-n)$ 不會是正交集合. 我們需要嚴建彰先生所導出的一個引理來幫助我們有較快的證明 [11]. (爲了方便讀者, 我們摘錄嚴先生的證明.)

Lemma 2.3. *Let* $\phi(x)$ *be an orthonormal scaling function and* $\psi(x)$ *be the associated orthonormal wavelet. If both* ϕ *and* ψ *are in* $\mathcal{C}^p$ *for some* $p > 0$, *then*

$$\langle \psi^{(r)}(x), \psi^{(s)}(x-n) \rangle = 2^{r+s} \langle \phi^{(r)}(x), \phi^{(s)}(x-2n) \rangle - \langle \phi^{(r)}(x), \phi^{(s)}(x-n) \rangle,$$

where $1 \le r, s \le p$.

Proof: *By* Theorem 2.2 *and the Parseval identity, we have*

$$\begin{aligned}
\langle \psi^{(r)}(x), \psi^{(s)}(x-n) \rangle &= \int (-i\omega)^r \hat{\psi}(\omega) \overline{e^{-i\omega n}(-i\omega)^s \hat{\psi}(\omega)}\, d\omega \\
&= \int (-1)^{(r-s)} (i\omega)^{r+s} |m_1(\frac{\omega}{2})|^2 e^{i\omega n} |\hat{\phi}(\frac{\omega}{2})|^2\, d\omega \\
&= \int (-1)^{(r-s)} (i\omega)^{r+s} (1 - |m_0(\frac{\omega}{2})|^2) e^{i\omega n} |\hat{\phi}(\frac{\omega}{2})|^2\, d\omega \\
&= \int (-1)^{(r-s)} (i\omega)^{r+s} e^{i\omega n} |\hat{\phi}(\frac{\omega}{2})|^2\, d\omega \\
&\qquad - \int (-1)^{(r-s)} (i\omega)^{r+s} |m_0(\frac{\omega}{2})|^2 e^{i\omega n} |\hat{\phi}(\frac{\omega}{2})|^2\, d\omega \\
&= 2^{r+s} \langle \phi^{(r)}(x), \phi^{(s)}(x-2n) \rangle - \langle \phi^{(r)}(x), \phi^{(s)}(x-n) \rangle.
\end{aligned}$$

This completes the proof. $\square$

Proposition 2.6. *Let* $\phi(x)$ *be an orthonormal scaling function, and* $\Phi(x)$ *is the elevation of* $\phi(x)$. *Then* $\Phi(x-n)$ *is not orthogonal.*

Proof: 若 $\Phi(x-n)$ 是正交集合. 我們可以調整 $\Phi(x)$ 的規模使 $\Phi(x-n)$ 是單位正交. 從 $\Phi(x)$ 的定義得知 Φ 和 Ψ 都屬於 $\mathcal{C}^1$. 我們應用上面的引理, $r = s = 1$ 的情形, 我們有

$$\begin{aligned}
\delta_{0,n} &= \langle \Psi'(x), \Psi'(x-n) \rangle \\
&= 4\langle \Phi'(x), \Phi'(x-2n) \rangle - \langle \Phi'(x), \Phi'(x-n) \rangle \\
&= \langle \Phi'(x), \Phi'(x-n) \rangle.
\end{aligned}$$

但是我們又可以很快的檢查

$$\langle \Phi'(x), \Phi'(x-n)\rangle = \begin{cases} 2 & \text{if } n = 0, \\ -1 & \text{if } n = 1 \text{ or } -1, \\ 0 & \text{otherwise.} \end{cases}$$

我們得到矛盾的結果, 所以 $\Phi(x-n)$ 不會是正交集合. □

第四節 雙正交系統的濾波係數

令 $\phi(x)$ 是正交自格函數, $\psi(x)$ 是相對應的正交凌波函數, 且 $(c_n)_{n=0}^{N-1}$ 是非零的自格係數. 若 $\phi(x)$ 有 p 階的逼近度, 則 $N = 2p$. 令 $\Phi(x)$ 和 $\tilde{\Phi}(x)$ 如同上一節所述, (h_n), $(\tilde{h}_n)$ 分別爲它們的自格係數. 從 (2.3), 我們可以導出下列演算法:

Algorithm 1.

```
$h_0 = \frac{c_0}{2}$, $h_N = \frac{c_{N-1}}{2}$
FOR $i = 1$ TO $N-1$
    $h_i = \frac{c_{i-1} + c_i}{2}$
END
```

對於 $\tilde{h}_n$, 我們比較 $m_0^{\tilde{\Phi}}$ 與 m_0^{ϕ} 的係數. 請參照 (2.10). 在實際應用上, 我們假設 $m_0^{\phi}(\omega)$ 有 $(1+e^{-i\omega})/2$ 的因子. 下列演算法將從 (c_n) 導出 $(\tilde{h}_n)$.

Algorithm 2.

```
$\tilde{h}_{-1} = 0$
FOR $i = 0$ TO $N-2$
    $\tilde{h}_i = 2c_i - \tilde{h}_{i-1}$
END
```

從 (2.8) 我們觀察到 Ψ 的消散數會比 ψ 的消散數多一. 反過來, 由 (2.10), $\tilde{\Psi}$ 的消散數會比 ψ 的消散數少一. 由 (2.5), 我們也看到 Φ 比 ϕ 還平滑; 從 (2.9), 我們則看到 $\tilde{\Phi}$ 會比 ϕ 來的不平滑. 這種一邊性質變好另一邊性質變爛的均衡情形是我們常常見到的. $\tilde{\Phi}$ 的自格係數 $(\tilde{h}_n)$ 也有類似 Proposition 2.2 的性質.

Proposition 2.7. *Let* $\phi(x)$ *be defined as in* (1.1) *and* c_n *satisfy the condition* (A) *in* (2.1). *If* $\tilde{\Phi}(x)$ *is defined by* $\chi_{[-1,0]} * \tilde{\Phi} = \phi$, *then the scaling coefficients* $(\tilde{h}_n)$ *of* $\tilde{\Phi}(x)$ *satisfy*

$$\sum_{n=0}^{2p-2} (-1)^n n^m \tilde{h}_n = 0, \text{ where } m = 0, \ldots, p-2 \quad (0^0 := 1).$$

Proof: 由 Algorithm 2 以及 $\sum(-1)^n c_n = 0$, 我們有

$$\tilde{h}_n = 2 \sum_{k=n+1}^{2p-1} (-1)^{k-n-1} c_k, \quad n = 0, \ldots, 2p-2.$$

從

$$\sum_{n=0}^{2p-1} n^m \sum_{k=n}^{2p-1} (-1)^k c_k = \sum_{n=0}^{2p-1} \left(\sum_{k=0}^{n} k^m \right) (-1)^n c_n,$$

我們尚且知道 $\sum_{k=1}^{n} k^m$ 是一個 n 的 $m+1$ 階的多項式. 再從 $(\tilde{h}_n)$ 滿足 (A), 我們得到

$$\sum_{n=1}^{2p-1} n^m \sum_{k=n}^{2p-1} (-1)^k c_k = 0, \text{ for } m = 0, \ldots, p-2.$$

因此, 對於 $0 \le m \le p-2$,

$$\begin{aligned}
\sum_{n=0}^{2p-2} (-1)^n n^m \tilde{h}_n &= 2 \sum_{n=0}^{2p-2} (-1)^n n^m \sum_{i=n+1}^{2p-1} (-1)^{i-n-1} c_i \\
&= -2 \sum_{n=1}^{2p-1} (n-1)^m \sum_{i=n}^{2p-1} (-1)^i c_i \\
&= -2 \sum_{n=1}^{2p-1} \left(\sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell n^\ell \right) \sum_{i=n}^{2p-1} (-1)^i c_i \\
&= -2 \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell \left(\sum_{n=1}^{2p-1} n^\ell \sum_{i=n}^{2p-1} (-1)^i c_i \right) \\
&= 0.
\end{aligned}$$

我們完成此一證明. □

索引

PART TWO

1 Introduction

Section 1 Motivation

Let $\phi(x)$ be a scaling function with a finite two-scale relation

$$\phi(x) = \sum_{n=0}^{N} c_n \phi(2x-n).$$

If $\phi(x)$ is compactly supported, then the support of ϕ is contained in $[0, N]$. The number of nonzero scaling coefficients determine the support of the scaling function. There is a simple way to see it. If $\phi(x)$ has a compact support, say $\operatorname{supp}\phi(x) = [a, b]$, then

$$\operatorname{supp}\phi(2x-n) = [\frac{a+n}{2}, \frac{b+n}{2}].$$

By the finite two-scale relation, the support of the right hand side is $[\frac{a}{2}, \frac{b+N}{2}]$. Compare with $[a, b]$, we see that $\operatorname{supp}\phi = [0, N]$.

It is obvious that the derivative of a scaling function still satisfies a scaling equation. The derivative of $\phi(x)$, if exists, satisfies $\phi'(x) = \sum_{n=0}^{N} 2c_n\phi'(2x-n)$. And $\phi'(x)$ still has a compact support, because the number of the nonzero scaling coefficients is unchanged. However, ϕ' fails to be a low pass filter, because

$$\hat{\phi}'(0) = \int_{-\infty}^{\infty} \phi'(x)dx = 0.$$

In particular, if we choose $\phi(x)$ to be the scaling function of Daubechies [4]. The linear combination of the translations of ϕ can express polynomials up to certain degree. For instance,

$$\sum_{n\in\mathbb{Z}} \phi(x-n) = 1, \ \forall x \in \mathbb{R}.$$

So $\sum \phi'(x-n) = 0$ and $\{\phi'(x-n)\}_{n\in\mathbb{Z}}$ is linearly dependent. It can not be a basis. On the other hand, the primitive of the scaling function has infinite support. Since

$$\int_0^N \phi(x)\,dx = 1,$$

the primitive of $\phi(x)$,

$$F(x) = \int_{-\infty}^x \phi(t)\,dt = 1, \quad \forall x \geq N.$$

Furthermore $F(x-n)$ cannot express the constant functions. For if it could, the set $\phi(x-n)$ is linearly dependent, which is contradictory.

By the above arguments, we can not derive a new scaling function from a given scaling function by either differentiation or integration. In this paper, we provide a scheme to derive a new scaling function from a given scaling function. The idea comes from the article of Xu and Shann [10]. We modify the privious definition of $F(x)$ to make it a new function with compact support, that is,

$$\Phi(x) = \int_{-\infty}^x \phi(t) - \phi(t-1)\,dt.$$

Then $\Phi(x)$ is a function similar to the primitive of $\phi(x)$, and surely $\Phi(x)$ is compactly supported. We will show that Φ is a scaling function and we will present many properties between Φ and ϕ.

Section 2 Notation

Let $\Omega \subset \mathbb{R}$ be a domain. Let $L^p(\Omega)$ be the Banach space of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f(x)|^p\,dx < \infty.$$

Let $\langle \cdot,\cdot \rangle$ denote the inner product of $L^2(\mathbb{R})$, namely,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx.$$

(1) If $\phi(x)$ satisfies the equation

$$\phi(x) = \sum_{n\in\mathbb{Z}} c_n \phi(2x-n), \tag{1.1}$$

we say that ϕ satisfies a *scaling equation*, and (c_n) is called the *scaling coefficients*. When we say that $\phi(x)$ is a *scaling function*, we mean that ϕ satisfies a scaling equation and the linear combinations of $\phi(x-n)$ can express polynomials up to a certain degree. In particular, if c_0, c_1, …, c_N are the only nonzero scaling coefficients, we say that ϕ satisfies a *finite scaling equation.* To be more precise, if $\phi(x-n)$ is an orthonormal set of functions, we call $\phi(x)$ an orthonormal scaling function [4]. In this paper, we always consider finite scaling equations, unless expressed otherwise.

Definition 1.1. *A multiresolution analysis is a sequence* (V_j) *of A subspaces of* $L^2(\mathbb{R})$ *such that*

(1) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$,
(2) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ *and* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(3) $f(x) \in V_j \iff f(2x) \in V_{j+1}$,
(4) $f(x) \in V_0 \iff f(x-n) \in V_0 \quad \forall n \in \mathbb{Z}$.

Let $\phi(x)$ be a scaling function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, define

$$V_j = \operatorname{span}\{\sqrt{2^j}\phi(2^j x - n) \mid n \in \mathbb{Z}\} \quad \forall j \in \mathbb{Z}. \tag{1.2}$$

If $\hat{\phi}(0) = 1$ and $\{\sqrt{2^j}\phi(2^j x - n) \mid n \in \mathbb{Z}\}$ is a Riesz basis of V_j, then by Proposition 5.3.1 and Proposition 5.3.2 of [5] the sequence (V_j) automatically forms a multiresolution analysis. We say that $\phi(x)$ generates a multiresolution analysis (MRA) (V_j) of $L^2(\mathbb{R})$. Let V_j be defined in (1.2), we define W_j to be the complement of V_j in V_{j+1}, that is,

$$V_{j+1} = V_j + W_j.$$

If W_j can be generated by a single function, say $\psi(x)$, in the sense that

$$W_j = \operatorname{span}\{\sqrt{2^j}\psi(2^j x - n) \mid n \in \mathbb{Z}\}. \tag{1.3}$$

And if $\psi(x)$ is at least exponentially decay, then we call the function $\psi(x)$ a *wavelet* associated with $\phi(x)$. To be more precise, if $\psi(x-n)$ is an orthonormal set of functions, we call $\psi(x)$ an orthonormal wavelet [4]. Otherwise, if W_j is the orthogonal complement of V_j in V_{j+1}, that is,

$$V_{j+1} = V_j \oplus W_j,$$

but W_j is not an orthogonal set, we call $\psi(x)$ a semi-orthogonal wavelet [2].

If $\psi(x)$ is neither an orthonormal nor a semi-orthogonal wavelet, we could find another pair of functions $\{\tilde{\phi}, \tilde{\psi}\}$ such that $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ forms a system of *biorthogonal* wavelets [3] in the following sense. Let $\tilde{\phi}(x)$ be another scaling function which generates another multiresolution analysis $(\tilde{V}_j)$ of $L^2(\mathbb{R})$. Define $\tilde{W}_j$ to be the complement of $\tilde{V}_j$ in $\tilde{V}_{j+1}$, as we did for W_j. And $\tilde{W}_k$ can be generated by a single function, say $\tilde{\psi}(x)$, in the sense similar to (1.3). Furthermore, we can design $\tilde{\phi}$ and $\tilde{\psi}$ such that

$$\langle \tilde{\phi}(x), \psi(x-n) \rangle = \langle \tilde{\psi}(x), \phi(x-n) \rangle = 0 \quad \forall n \in \mathbb{Z}, \tag{1.4}$$

and

$$\langle \tilde{\phi}(x), \phi(x-n) \rangle = \langle \tilde{\psi}(x), \psi(x-n) \rangle = \delta_{0n} \quad \forall n \in \mathbb{Z}. \tag{1.5}$$

Here δ_{0n} is the Direc delta notation. In other words,

$$V_j \perp \tilde{W}_j \, , \, \tilde{V}_j \perp W_j \quad \forall j \in \mathbb{Z}.$$

(2) Let $*$ be the convolution operator defined by

$$f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy \qquad \text{for almost all } x, \tag{1.6}$$

where f and g are both integrable functions (f and $g \in L^1(\mathbb{R})$). It is clear that $f * g$ is still integrable. In fact, since

$$\begin{aligned} \int |f * g(x)|\, dx &\leq \int\int |f(x-y)|\, |g(y)|\, dy\, dx \\ &= \int |g(y)| \left[\int |f(x-y)|\, dx \right] dy \\ &= \int |g(y)| \left[\int |f(x)|\, dx \right] dy, \end{aligned}$$

$$\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1.$$

Similarly, $f * g \in L^2(\mathbb{R})$ if both f and $g \in L^2(\mathbb{R})$. It is also well-known that the convolution is commutative, $f * g = g * f$; distributive, $f * (g + h) = f * g + f * h$; and associative, $f * (g * h) = (f * g) * h$.

Similarly, we can define the discrete convolution for sequences (x_n) and (y_n) in ℓ^2:

$$(x * y)_n = \sum_k x_{n-k} y_k. \tag{1.7}$$

(3) Let $\hat{f}(\omega)$ be the Fourier transform of $f(x)$. That is,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \langle f(x), e^{i\omega x} \rangle.$$

We take it without proof that the Fourier transform exists for all $f(x)$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. And the inverse Fourier transform is defined by

$$\check{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)\, e^{ix\omega}\, d\omega.$$

If $f(x)$ is continuous or at its discontinuity it satisfies $f(x) = \frac{1}{2}[f(x-) + f(x+)]$, then

$$f(x) = \left(\hat{f}(\omega)\right)^{\vee}(x).$$

We first recall the relation between convolution and the Fourier transform. Assume that both f and $g \in L^1(\mathbb{R})$

$$\begin{aligned}
\widehat{f * g} &= \int_{-\infty}^{\infty} f * g(x)\, e^{-i\omega x}\, dx \\
&= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} f(x-y)\, g(y)\, dy\, e^{-i\omega x}\, dx \\
&= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} f(x-y)\, g(y)\, dy\, e^{-i\omega(x-y)}\, e^{-i\omega y}\, dx \\
&= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} f(x-y)\, g(y) e^{-i\omega(x-y)}\, e^{-i\omega y}\, dx\, dy \\
&= \int_{-\infty}^{\infty} \hat{f}(\omega)\, g(y)\, e^{-i\omega y}\, dy \\
&= \hat{f}(\omega)\, \hat{g}(\omega).
\end{aligned}$$

That is, the Fourier transform of the convolution of the functions f and g is equivalent to the product of the Fourier transforms of f and g.

Sometimes we use $\mathcal{F}$ to denote the Fourier transform:

$$\mathcal{F}\{f(x)\}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx.$$

With this notation, we observe the relation between dilation/translation and the Fourier transform.

$$\begin{aligned}\mathcal{F}\{f(ax-k)\}(\omega) &= \int_{-\infty}^{\infty} f(ax-k)\, e^{-i\omega x}dx \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega\frac{at+k}{a}}\; e^{-i\frac{k}{a}\omega}\, dx \\ &= \frac{1}{a}e^{-i\frac{k}{a}\omega}\int_{-\infty}^{\infty} f(t)\, e^{-i\frac{\omega}{a}t}dt \\ &= \frac{1}{a}e^{-i\frac{k}{a}\omega}\hat{f}(\frac{\omega}{a}) \\ &= \frac{1}{a}e^{-i\frac{k}{a}\omega}\mathcal{F}\{f(x)\}(\frac{\omega}{a}).\end{aligned}$$

For instance,

$$\mathcal{F}\{\sqrt{2^j}\phi(2^jx-n)\}(\omega) = \frac{1}{\sqrt{2^j}}e^{-in\frac{\omega}{2^{-j}}}\hat{\phi}(\frac{\omega}{2^j}).$$

(4) Let $\phi(x)$ and (c_n) be defined as in (1.1). Then the Fourier transform of $\phi(x)$ is

$$\begin{aligned}\hat{\phi}(\omega) &= \int_{-\infty}^{\infty} \phi(x)\, e^{-i\omega x}\, dx \\ &= \int_{-\infty}^{\infty} (\sum_{n\in\mathbb{Z}} c_n\phi(2x-n))\, e^{-i\omega x}\, dx \\ &= \sum_{n\in\mathbb{Z}} c_n \int_{-\infty}^{\infty} \phi(2x-n)\, e^{-i\omega x}\, dx \\ &= \sum_{n\in\mathbb{Z}} c_n\mathcal{F}\{\phi(2x-n)\}(\omega) \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}} c_n\, e^{-i\frac{\omega}{2}n}\hat{\phi}(\frac{\omega}{2}).\end{aligned}$$

We define a 2π-periodic function $m_0^{\phi}(\omega)$,the characteristic function of $\phi(x)$, by

$$m_0^{\phi}(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} c_n\, e^{-i\omega n}.$$

Then

$$\hat{\phi}(\omega) = m_0^{\phi}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}). \tag{1.8}$$

Let ψ be the wavelet associated with ϕ, as defined in (1.3). Then ψ should be of the form

$$\psi(x) = \sum_{n\in\mathbb{Z}} d_n\phi(2x-n). \tag{1.9}$$

(d_n) is called the *wavelet coefficients* of ψ. We will see that (d_n) can be derived from the scaling coefficients (c_n). For instance, in case of orthonormal wavelets,

$$d_n = (-1)^n c_{1-n}.$$

Similarly, take the Fourier transform of ψ, we get

$$\hat{\psi}(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} d_n\, e^{-i\frac{\omega}{2}n}\hat{\phi}(\frac{\omega}{2}).$$

Now we define

$$m_1^{\phi}(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} d_n\, e^{-i\omega n}.$$

$m_1^{\phi}(\omega)$ is called the characteristic function of $\psi(x)$ associated with $\phi(x)$. That is,

$$\hat{\psi}(\omega) = m_1^{\phi}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}). \tag{1.10}$$

When there is no ambiguity, we will drop the index ϕ and use m_0 and m_1.

Let p be the number of *vanishing moments* of the wavelets, that is,

$$\int x^m \psi(x)\, dx = 0 \text{ for } 0 \le m < p.$$

Because

$$\hat{\psi}^{(m)}(0) = 0 \iff \int x^m \psi(x)\, dx = 0.$$

By (1.10) and the fact that $\hat{\phi}(0) \neq 0$, it can be shown that 0 is a root of multiplicity p for $m_1(\omega)$. We can write

$$m_1(\omega) = \left(\frac{1-e^{-i\omega}}{2}\right)^p Q(\omega),$$

where $Q(0) \neq 0$.

(5)A function $S(\omega)$ is said to be a *Laurent series* if $S(\omega)$ is of the form

$$S(\omega) = \sum_n s_n e^{-i\omega n}.$$

We define a new class $\mathcal{A}$ by

$$\mathcal{A} = \{S : S(\omega) = \sum s_n e^{-i\omega n}, \quad (s_n) \in \ell^1 \text{ and } S(\omega) \neq 0\}. \tag{1.11}$$

Since the discerte convolution of two ℓ^1-sequence is still a sequence in ℓ^1, it is clear that $\mathcal{A}$ is an algebra. Moreover, if $S(\omega) \in \mathcal{A}$, then $\frac{1}{S(\omega)} \in \mathcal{A}$, too.

2 Fundamentals

Section 1 Scaling functions

Let ϕ be defined as in (1.1). By (1.8), if $\hat{\phi}(0) \neq 0$, we must have $\sum c_n = 2$. This is essentially the only interesting case for us. Since we are interested in the low pass filters, we need $\hat{\phi}(0) \neq 0$. Let us now normalize $\hat{\phi}(0) = 1$. By iterating (1.8) we get

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} m_0(2^{-j}\omega),$$

where the infinite product must converge uniformly on every compact subset of $\mathbb{R}$. It can be proved by the facts that $m_0(0) = 1$ and $m_0(\omega)$ is continuous. Whether ϕ belongs to $L^1(\mathbb{R})$ or not, if the scaling coefficients sequence (c_n) is finite and $\sum c_n = 2$, then ϕ is a distribution or a generalized function.

Definition 2.1. *A distribution is a mapping* $F : C_0^{(\infty)} \to \mathbb{C}$ *that satisfies the following conditions:*

(1)Linearity:$F(c_1\phi_1 + c_2\phi_2) = c_1F(\phi_1) + c_2F(\phi_2)$ for all $\phi_1, \phi_2 \in C_0^{(\infty)}$ and all $c_1, c_2 \in \mathbb{C}$.
(2)Continuity:Suppose (ϕ_k) is a sequence in $C_0^{(\infty)}$ such that $\text{supp}(\phi_k)$ is contained in a fixed bounded set D for all k, and suppose that the functions ϕ_k and all their derivatives $\partial^\alpha \phi_k$ converge uniformly to zero as $k \to \infty$. Then $F(\phi_k) \to 0$.

Theorem 2.1. *Let* $c_0, \ldots, c_N \in \mathbb{R}$ *with* $\sum c_n = 2$. *Then there is a unique compactly supported distribution* ϕ *satisfies* (1.1) *with* $\hat{\phi}(0) = 1$. *The support of* ϕ *is contained in* $[0, N]$.

Proof: Define the sequence of the distribution (ϕ_n) by

$$\hat{\phi}_n(\omega) = \prod_{j=1}^{n} m_0(2^{-j}\omega).$$

Since $m_0(0) = 1$, we have $|m_0(\omega) - 1| \leq C|\omega|$ for some positive constant C. This ensures that $\hat{\phi}_n(\omega)$ converges uniformly on every compact subset of $\mathbb{R}$ toward an entire function ϕ as $n \to \infty$. In particular we have, for all n,

$$\sup_{|\omega|\leq 2} |\hat{\phi}_n(\omega)| \leq M < \infty,$$

and there are $R \geq 0$ and $B \geq 1$ such that

$$|m_0(\omega)| \leq Be^{R|\omega|}.$$

Then we have the fundamental estimate:

$$|\hat{\phi}_n(\omega)| \leq M(1 + |\omega|)^{\log_2 B} e^{R|\omega|}$$

by considering the cases $2^m \leq |\omega| \leq 2^{m+1}$, $\forall m \in \mathbb{Z}$, and $|\omega| \leq 2$ separately. Letting $n \to \infty$, we see that the same inequality holds for ϕ. Then by the *Paley-Wiener Theorem* [7, p.184], ϕ is the Fourier transform of a compactly supported distribution.
□

In this paper, when we say that $\phi(x)$ is a scaling function, we mean that $\phi(x)$ satisfies a scaling relation and the linear combinations of $\{\phi(\cdot - n)\}$ can express polynomials up to a certain degree. The power of approximation depends on the scaling coefficients. We have the following proposition.

Proposition 2.1. *Let* $\phi(x)$ *be defined as in* (1.1). *If the scaling coefficients* (c_n) *satisfy*

$$\sum_{n\in\mathbb{Z}} (-1)^n n^m c_n = 0, \tag{2.1}$$

for integer $0 \leq m \leq p$, *and the scaling function satisfies*

$$\sum_{n\in\mathbb{Z}} \phi(x - n) = 1 \quad \forall x \in \mathbb{R},$$

then $\sum_{n\in\mathbb{Z}} n^m \phi(x - n)$ *is a polynomial of degree* m.

Proof: Identity (2.1) is equivalent to $m_0(\omega)$ having a root of multiplicity p at $\omega = \pi$, which, by (1.8), implies that

$$i^m \hat{\phi}^{(m)}(2k\pi) = \delta_{0,k} \int x^m \phi(x)\, dx.$$

By Poisson's summation formula, it follows that

$$\sum_{n\in\mathbb{Z}} (x-n)^m \phi(x-n) = \int x^m \phi(x)\, dx. \qquad \square$$

Section 2 Orthonormal wavelets

For a given scaling function $\phi(x)$, if $\phi(x)$ generates a multiresolution analysis, then we can derive a wavelet from the scaling function. In the orthonormal case, let $\phi(x)$ be defined as in (1.1), and $\phi(x-n)$ be an orthonormal basis of V_0. The orthonormality of the $\phi(x-n)$ leads to special properties for $m_0(\omega)$. We start with

$$\begin{aligned}\delta_{k,0} &= \int \phi(x)\overline{\phi(x-k)}dx \\ &= \int \overline{\hat{\phi}(\omega)}\hat{\phi}(\omega)e^{i\omega k}d\omega \\ &= \int |\hat{\phi}(\omega)|^2 e^{ik\omega}d\omega \\ &= \int_0^{2\pi} [\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2] e^{ik\omega} d\omega.\end{aligned}$$

Because k is arbitrary, this implies

$$\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 = (2\pi)^{-1} \quad \text{a.e.} \tag{2.2}$$

Substituting (1.8) into (2.2) leads to

$$\sum_{\ell\in\mathbb{Z}} |m_0(\omega + \pi\ell)|^2 |\hat{\phi}(\omega + \pi\ell)|^2 = (2\pi)^{-1} \quad \text{a.e.}; \tag{2.3}$$

splitting the index into even and odd ℓ, using the periodicity of $m_0(\omega)$ and applying (2.2) once more gives

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1 \quad \text{a.e.} \tag{2.4}$$

This is an important property of the orthonormality of the scaling function.

Let us now characterize W_0. $f \in W_0$ is equivalent to $f \in V_1$ and $f \perp V_0$. Since $f \in V_1$, we have

$$f = \sum_{n\in\mathbb{Z}} f_n \phi(2x-n),$$

with $f_n = \langle f(x), \phi(2x-n)\rangle$. This implies

$$\hat{f}(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} f_n e^{-in\frac{\omega}{2}}\hat{\phi}(\frac{\omega}{2}) = m_f(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}), \tag{2.5}$$

where

$$m_f(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} f_n e^{-in\omega}; \tag{2.6}$$

$m_f(\omega)$ is a 2π-periodic function in $L^2([0,2\pi])$; convegence in (2.6) holds pointwise a.e.. The constraint $f \perp V_0$ implies $f \perp \phi_{0,k}$ for all k, i.e.,

$$\int \hat{f}(\omega)\overline{\hat{\phi}(\omega)}e^{-ik\omega}\,d\omega = 0$$

or

$$\int_0^{2\pi}[\sum_{\ell\in\mathbb{Z}} \hat{f}(\omega+2\pi\ell)\overline{\hat{\phi}(\omega+2\pi\ell)}]e^{-ik\omega}\,d\omega = 0.$$

Hance

$$\sum_{\ell\in\mathbb{Z}} \hat{f}(\omega+2\pi\ell)\overline{\hat{\phi}(\omega+2\pi\ell)} = 0 \tag{2.7}$$

and the series converges absolutely in $L^1([-\pi,\pi])$. Substituting (1.8) and (2.5), regrouping the sums for odd and even ℓ, and using (2.2) leads to

$$m_f(\omega)\overline{m_0(\omega)} + m_f(\omega+\pi)\overline{m_0(\omega+\pi)} = 0 \quad \text{a.e.} \tag{2.8}$$

Since $m_0(\omega)$ and $m_0(\omega+\pi)$ cannot vanish together on a set of nonzero measure (because of (2.4)), this implies the existence of a 2π-periodic function $\lambda(\omega)$ so that

$$m_f(\omega) = \lambda(\omega)\overline{m_0(\omega+\pi)} \quad \text{a.e.} \tag{2.9}$$

and by (2.8)

$$\lambda(\omega) + \lambda(\omega+\pi) = 0 \quad \text{a.e.} \tag{2.10}$$

This last equation can be recast [5] as

$$\lambda(\omega) = e^{i\omega}\nu(2\omega), \tag{2.11}$$

where $\nu(\omega)$ is 2π-periodic. Substituting (2.11) and (2.9) into (2.5) gives

$$\hat{f}(\omega) = e^{i\frac{\omega}{2}}\overline{m_0(\frac{\omega}{2}+\pi)}\nu(\omega)\hat{\phi}(\frac{\omega}{2}). \tag{2.12}$$

We will show that the orthonormal wavelet is of the form

$$\hat{\psi}(\omega) = \pm e^{i\frac{\omega}{2}}\overline{m_0(\frac{\omega}{2}+\pi)}\hat{\phi}(\frac{\omega}{2}) \tag{2.13}$$

Disregarding convergence questions, (2.12) can be written formally as

$$\hat{f}(\omega) = \left(\sum_{k\in\mathbb{Z}} \nu_k e^{-ik\omega}\right)\hat{\psi}(\omega).$$

That is

$$f(x) = \sum_{k\in\mathbb{Z}} \nu_k \psi(x-k).$$

Proposition 2.2. *Let* $\phi(x)$ *be a scaling function such that* $\phi(x-n)$ *is an orthonormal basis of* V_0. *Define the wavelet* $\psi(x)$ *as in* (2.13), *then* $\psi(x-n)$ *is an orthonormal basis of* W_0.

Proof: First of all, the properties of $m_0(\omega)$ and $\hat{\phi}(\omega)$ ensure that (2.13) defines indeed an L^2-function $\in V_1$ and $\perp V_0$, so that $\psi(x) \in W_0$. By the orthonormality of $f(x-n)$,

$$\begin{aligned}
\delta_{0,k} &= \int |\hat{f}(\omega)|^2 e^{i\omega k}\, d\omega \\
&= \int |m_0(\frac{\omega}{2}+\pi)\hat{\phi}(\frac{\omega}{2})\nu(\omega)|^2 e^{i\omega k}\, d\omega \\
&= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} |m_0(\frac{\omega}{2}+(n+1)\pi)\hat{\phi}(\frac{\omega}{2}+n\pi)\nu(\omega)|^2 e^{i\omega k}\, d\omega.
\end{aligned}$$

Since k is arbitrary, we have

$$\sum_{n\in\mathbb{Z}} |m_0(\frac{\omega}{2}+(n+1)\pi)\hat{\phi}(\frac{\omega}{2}+n\pi)\nu(\omega)|^2 = (2\pi)^{-1}.$$

By using (2.4), we have

$$|\nu(\omega)|^2\left\{\sum_{n\in\mathbb{Z}}|\hat{\phi}(\frac{\omega}{2}+n\pi)|^2-\sum_{n\in\mathbb{Z}}|m_0(\frac{\omega}{2}+n\pi)|^2|\hat{\phi}(\frac{\omega}{2}+n\pi)|^2\right\}=(2\pi)^{-1}.$$

And by (2.2) and (2.3), $\nu(\omega)=\pm 1$ almost everywhere. Then $\psi(x)$ is of the form (2.13). In order to check that the $\psi(x-n)$ are indeed a basis for W_0, it suffices to check that for all $f(x)\in W_0$, $f(x)$ can be written as

$$f(x)=\sum_{n\in\mathbb{Z}}\gamma_n\psi(x-n),$$

with $\sum|\gamma_n|^2<\infty$. For if right,

$$\hat{f}(\omega)=\left(\sum_{n\in\mathbb{Z}}\gamma_n e^{-i\omega n}\right)\hat{\psi}(\omega).$$

Let $\gamma(\omega)=\sum\gamma_n e^{-i\omega n}$. We have

$$\{\gamma_n\}\in\ell^2 \iff \gamma(\omega)\in L^2([0,2\pi]).$$

That is

$$\hat{f}(\omega)=\gamma(\omega)\hat{\psi}(\omega), \tag{2.14}$$

where $\gamma(\omega)$ is 2π-periodic and belongs $L^2([0,2\pi])$. Let us back to (2.12). We have $\hat{f}(\omega)=\gamma(\omega)\hat{\psi}(\omega)$, with $\int_0^{2\pi}|\gamma(\omega)|^2d\omega=2\int_0^{\pi}|\lambda(\omega)|^2d\omega$. By (2.6),

$$\int_0^{2\pi}|m_f(\xi)|^2d\xi=\pi\sum_{n\in\mathbb{Z}}|f_n|^2=\pi\|f\|^2<\infty.$$

On the other hand, by (2.9),

$$\begin{aligned}\int_0^{2\pi}|m_f(\xi)|^2d\xi&=\int_0^{2\pi}|\lambda(\xi)|^2|m_0(\xi+\pi)|^2d\xi\\&=\int_0^{\pi}|\lambda(\xi)|^2[|m_0(\xi+\pi)|^2+|m_0(\xi)|^2]d\xi &&\text{By (2.10)}\\&=\int_0^{\pi}|\lambda(\xi)|^2\,d\xi. &&\text{By (2.4)}\end{aligned}$$

Hence $\int_0^{2\pi}|\gamma(\xi)|^2d\xi=2\pi\|f\|^2<\infty$, and $f(x)$ is of the form (2.14) with square integrable 2π-periodic $\gamma(\omega)$. We have proved that $\psi(x-n)$ is a basis of W_0. $\square$

Note that the wavelet is not determined uniquely by the given multiresolution analysis, but the orthonormal wavelet $\psi(x)$ is uniqluy determined (up to sign). For the above case, $m_1(\omega)$ is defined as $e^{i\omega/2}\overline{m_0(\omega/2+\pi)}$. Then the wavelet coefficients d_n become $d_n = (-1)^n c_{1-n}$.

If $\phi(x-n)$ is not an orthonormal set, there are two important ways for making wavelets. One produces semi-orthogonal wavelets, and the other produces biorthogonal wavelets.

Section 3 Semi-orthogonal wavelets

By the semi-orthogonality, $V_1 = V_0 \oplus W_0$. If $\psi(x-n)$ is a basis of W_0, then for all $n \in \mathbb{Z}$, We have

$$\begin{aligned}
\langle \phi(x), \psi(x-n)\rangle &= \int \hat{\phi}(\omega)\overline{\hat{\psi}(\omega)}e^{i\omega n}\,d\omega \\
&= \int e^{i\omega n} m_0(\frac{\omega}{2})\overline{m_1(\frac{\omega}{2})}|\hat{\phi}(\frac{\omega}{2})|^2\,d\omega \\
&= \int_0^{4\pi} e^{i\omega n} m_0(\frac{\omega}{2})\overline{m_1(\frac{\omega}{2})}\sum_k |\hat{\phi}(\frac{\omega}{2}+2\pi k)|^2\,d\omega \\
&= \int_0^{2\pi} e^{i\omega n}\Big(m_0(\frac{\omega}{2})\overline{m_1(\frac{\omega}{2})}\sum_k |\hat{\phi}(\frac{\omega}{2}+2\pi k)|^2 \\
&\qquad + m_0(\frac{\omega}{2}+\pi)\overline{m_1(\frac{\omega}{2}+\pi)}\sum_k |\hat{\phi}(\frac{\omega}{2}+2\pi k+\pi)|^2\Big)\,d\omega.
\end{aligned}$$

Since n is arbitrary, we have

$$m_0(\omega)\overline{m_1(\omega)}\Gamma(\omega) + m_0(\omega+\pi)\overline{m_1(\omega+\pi)}\Gamma(\omega+\pi) = 0 \quad \text{a.e.,} \tag{2.15}$$

where $\Gamma(\omega) := \sum |\hat{\phi}(\omega+2\pi k)|^2$. By [2] we will see that if $\phi(x)$ has compact support, $\Gamma(\omega) = \sum \gamma_n e^{-i\omega n}$ with finite sequence (γ_n). And from (2.15) we get

$$m_1(\omega) = e^{-i\omega}\overline{m_0(\omega+\pi)}\Gamma(\omega+\pi)s(2\omega) \quad \text{for some } s \in \mathcal{A}, \tag{2.16}$$

where $\mathcal{A}$ is defined as in (1.11). We would show that such $\psi(x-n)$ is indeed a Riesz basis of W_0.

Proposition 2.3. *Let ϕ be defined as in* (1.1) *and $m_1(\omega)$ be defined as above. If $\Gamma(\omega)$ is also in $\mathcal{A}$, then ψ is a semi-orthogonal wavelet of ϕ.*

Proof: Since $\hat{\psi}(\omega) = m_1(\frac{\omega}{2})\,\hat{\phi}(\frac{\omega}{2})$ and by the periodicity of $m_0(\omega)$ and $\Gamma(\omega)$, we have

$$\begin{aligned}\sum_n |\hat{\psi}(\omega+2\pi n)|^2 &= \sum_n |m_1(\frac{\omega}{2}+\pi n)\hat{\phi}(\frac{\omega}{2}+\pi n)|^2\\ &= \sum_n |e^{-i(\frac{\omega}{2}+\pi n)} m_0(\frac{\omega}{2}+\pi n+\pi)\\ &\qquad\times \Gamma(\frac{\omega}{2}+\pi n+\pi)s(\omega)\hat{\phi}(\frac{\omega}{2}+\pi n)|^2\\ &= |s(\omega)e^{-i\frac{\omega}{2}}|^2\Gamma(\frac{\omega}{2}+\pi)\Gamma(\frac{\omega}{2})\\ &\qquad\times\left(|m_0(\frac{\omega}{2}+\pi)|^2\Gamma(\frac{\omega}{2}+\pi)+|m_0(\frac{\omega}{2})|^2\Gamma(\frac{\omega}{2})\right).\end{aligned}$$

By [2]

$$|m_0(\frac{\omega}{2}+\pi)|^2\Gamma(\frac{\omega}{2}+\pi)+|m_0(\frac{\omega}{2})|^2\Gamma(\frac{\omega}{2}) = \Gamma(\omega), \tag{2.17}$$

then we have

$$\sum_n |\hat{\psi}(\omega+2\pi n)|^2 = |s(\omega)e^{-i\frac{\omega}{2}}|^2\Gamma(\frac{\omega}{2}+\pi)\Gamma(\frac{\omega}{2})\Gamma(\omega).$$

Because $\phi(x-n)$ is a Riesz basis of V_0, $\Gamma(\omega)$ is bounded away from zero a.e.. Then there are $0 < A \le B < \infty$ such that

$$A \le \sum_n |\hat{\phi}(\omega+2\pi n)|^2 \le B \quad \text{a.e..}$$

Since $s(\omega)$ belongs to $\mathcal{A}$, let $L = \min s(\omega) > 0$ and $U = \max s(\omega) < \infty$, we have

$$A^3L^2 \le \sum_n |\hat{\psi}(\omega+2\pi n)|^2 \le B^3U^2 \quad \text{a.e..}$$

That is, $\psi(x-n)$ is a Riesz basis of W_0.

Next we claim that $V_1 = V_0 + W_0$. That is, there must be two sequences (a_n) and (b_n) such that

$$\phi(2x-\ell) = \sum_n \{a_{\ell-2n}\phi(x-n) + b_{\ell-2n}\psi(x-n)\}. \tag{2.18}$$

Now, we should derive the sequences (a_n) and (b_n) from $m_0(\omega)$ and $m_1(\omega)$. By taking the Fourier transform of the decomposition relation (2.18), we have

$$\begin{aligned}\frac{1}{2}\hat{\phi}(\frac{\omega}{2})e^{-i\ell\frac{\omega}{2}} &= \sum_n a_{\ell-2n}e^{-i\omega n}\hat{\phi}(\omega) + \sum_n b_{\ell-2n}e^{-i\omega n}\hat{\psi}(\omega)\\ &= \sum_n a_{\ell-2n}e^{-i\omega n}m_0(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}) + \sum_n b_{\ell-2n}e^{-i\omega n}m_1(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}).\end{aligned}$$

That is

$$\Big(\sum_n a_{\ell-2n}e^{i(\ell-2n)\omega}\Big)m_0(\omega)+\Big(\sum_n b_{\ell-2n}e^{i(\ell-2n)\omega}\Big)m_1(\omega)\equiv\frac{1}{2}\quad\forall\ell\in\mathbb{Z}.\tag{2.19}$$

Define two 2π-periodic functions $\tilde{m}_0(\omega)$ and $\tilde{m}_1(\omega)$ by

$$\tilde{m}_0(\omega)=\sum_n a_n e^{-i\omega n}\quad\text{and}\quad\tilde{m}_1(\omega)=\sum_n b_n e^{-i\omega n}.$$

Splitting (2.19) into even and odd ℓ, we have

$$(\tilde{m}_0(\omega)+\tilde{m}_0(\omega+\pi))\overline{m_0(\omega)}+(\tilde{m}_1(\omega)+\tilde{m}_1(\omega+\pi))\overline{m_1(\omega)}=1\quad\text{for even }\ell$$

and

$$(\tilde{m}_0(\omega)-\tilde{m}_0(\omega+\pi))\overline{m_0(\omega)}+(\tilde{m}_1(\omega)-\tilde{m}_1(\omega+\pi))\overline{m_1(\omega)}=1\quad\text{for odd }\ell.$$

The above two equations lead to

$$\begin{aligned}\tilde{m}_0(\omega)\overline{m_0(\omega)}+\tilde{m}_1(\omega)\overline{m_1(\omega)}&=1\\ \tilde{m}_0(\omega)\overline{m_0(\omega+\pi)}+\tilde{m}_1(\omega)\overline{m_1(\omega+\pi)}&=0.\end{aligned}$$

Finally, we have

$$\begin{aligned}\tilde{m}_0(\omega)&=\frac{\overline{m_1(\omega+\pi)}}{\Delta_{\phi,\psi}(\omega)}\\ \tilde{m}_1(\omega)&=\frac{-\overline{m_0(\omega+\pi)}}{\Delta_{\phi,\psi}(\omega)},\end{aligned}$$

where $\Delta_{\phi,\psi}(\omega):=\overline{m_0(\omega)m_1(\omega+\pi)-m_0(\omega+\pi)m_1(\omega)}$. The condition that both $s(\omega)$ and $\Gamma(\omega)\in\mathcal{A}$ implies $\Delta_{\phi,\psi}(\omega)\in\mathcal{A}$, hence both $\tilde{m}_0(\omega)$ and $\tilde{m}_1(\omega)\in\mathcal{W}$. $\square$

When $\phi(x-n)$ is an orthonormal basis of V_0, then $\Gamma(\omega)\equiv1$. If we chose $s(\omega)\equiv1$, then $\sum|\hat{\psi}(\omega+2\pi n)|^2=1$ for all $\omega\in\mathbb{R}$. That is $\psi(x-n)$ is also an orthonormal basis of W_0. It is clear that the rate of decay of $\psi(x)$ is governed by the function $s(\omega)$. If the coefficients of $s(\omega)$ is finite, then the wavelet $\psi(x)$ is compactly supported.

By (2.16) and (2.17), we rewrite $\tilde{m}_0(\omega)$ to be ($s(\omega)$ has been cancelled)

$$\tilde{m}_0(\omega)=\frac{\overline{m_1(\omega+\pi)}}{\Delta_{\phi,\psi}(\omega)}=m_0(\omega)\frac{\Gamma(\omega)}{\Gamma(2\omega)}.\tag{2.20}$$

In general, $\Gamma(\omega)$ does not have a factor $\Gamma(2\omega)$, so $\tilde{\phi}$ and $\tilde{\psi}$ can not be compactly supported. Moreover, $\tilde{\phi}$ and ϕ have the following proposition. By this proposition, we will see that $V_0 \subset \tilde{V}_0$. Here $\tilde{V}_j$ is the multiresolution generated by $\tilde{\phi}(x)$. Although $\phi(x)$ is compactly supported and $\tilde{\phi}(x)$ has infinite support, $\phi(x)$ is a finite linear combination of $\tilde{\phi}(x-n)$.

Proposition 2.4. *Let* $\phi(x)$ *be defined as in* (1.1) *and* $\tilde{\phi}$ *is the semi-orthogonornal dual scaling function associated to* $\phi(x)$. *That is*

$$\langle \phi(x), \tilde{\phi}(x-n) \rangle = \delta_{0,n} \quad \forall n \in \mathbb{Z}.$$

Let γ_n *be defined by*

$$\gamma_n = \langle \phi(x), \phi(x-n) \rangle \quad \forall n \in \mathbb{Z}.$$

Then, when the support of $\phi(x)$ *is contained in* $[0, N]$, *we have*

$$\phi(x) = \sum_{n=-N+1}^{N-1} \gamma_n \tilde{\phi}(x-n). \tag{2.21}$$

Proof: By (2.20) and (1.8), we have

$$\Gamma(\omega)\hat{\tilde{\phi}}(\omega) = m_0(\frac{\omega}{2})\Gamma(\frac{\omega}{2})\hat{\tilde{\phi}}(\frac{\omega}{2}).$$

Then

$$\begin{aligned}\Gamma(\omega)\hat{\tilde{\phi}}(\omega) &= \lim_{k\to\infty}\left[\prod_{j=1}^{k} m_0(\frac{\omega}{2^k})\right]\Gamma(\frac{\omega}{2^k})\hat{\tilde{\phi}}(\frac{\omega}{2^k}) \\ &= \hat{\phi}(\omega).\end{aligned}$$

Since $\Gamma(\omega) = \sum_{n=-N+1}^{N-1} \gamma_n e^{-i\omega n}$, we have

$$\phi(x) = \sum_{n=-N+1}^{N-1} \gamma_n \tilde{\phi}(x-n).$$

This completes the proof. $\square$

We don't know whether $\tilde{V}_0 \subset V_0$ yet. Define a 2π-period function $\tilde{\Gamma}(\omega)$ for $\tilde{\phi}(x)$ as we did for $\phi(x)$. When $\phi(x)$ is B-spline, the coefficients of $\tilde{\Gamma}(\omega)$, $\tilde{\gamma}_n$, belongs to ℓ^1. Moreover, if $\phi(x)$ makes $\Gamma(\omega) \in \mathcal{A}$, then $\Gamma^{-1}(\omega)$ exists and belongs to $\mathcal{A}$. Fortunately, $\tilde{\Gamma}(\omega)$ is equal to $\Gamma^{-1}(\omega)$. Then the coefficients $(\tilde{\gamma}_n)$ of $\tilde{\Gamma}(\omega)$ belongs to ℓ^1. And we have the following lemma.

Lemma 2.1. *Let* $\phi(x)$ *be defined as in* (1.1), $\phi(x) \in L^2(R)$ *and* $\tilde{\phi}(x)$ *is the daul of* $\phi(x)$ *in the semi-orthogornal system. If* $(\tilde{\gamma}_n) \in \ell^1$, *then* $\tilde{\phi}(x) \in L^2(\mathbb{R})$.

Proof: When we replace $\tilde{\phi}(x)$ with $\phi(x)$, $\phi(x)$ plays a role as $\tilde{\phi}(x)$ in (2.20). Then

$$\tilde{\phi}(x) = \sum_n \tilde{\gamma}_n \phi(x-n).$$

By two scaling relation and $(\tilde{\gamma}_n) \in \ell^1$,

$$\begin{aligned}\int |\tilde{\phi}(x)|^2\, dx &= \int \tilde{\phi}(x)\overline{\tilde{\phi}(x)}\, dx \\ &= \int \Big(\sum_n \tilde{\gamma}_n \phi(x-n)\Big) \overline{\Big(\sum_m \tilde{\gamma}_m \phi(x-m)\Big)}\, dx \\ &= \sum_{n,m} \tilde{\gamma}_n \tilde{\gamma}_m \langle \phi(x-n), \phi(x-m)\rangle\end{aligned}$$

Since $\phi(x)$ is compactly supported, only finitely many of $\{\gamma_n\}$ are nonzero. Let $C = \max\{|\gamma_n|\}$, then

$$\begin{aligned}\int |\tilde{\phi}(x)|^2\, dx &\le C \sum_{n,m} \tilde{\gamma}_n \tilde{\gamma}_m \\ &\le C\Big(\sum_n |\tilde{\gamma}_n|\Big)\Big(\sum_m |\tilde{\gamma}_m|\Big) < \infty.\end{aligned}$$

□

By the previous lemma, if $\Gamma(\omega) \in \mathcal{A}$, we have $\tilde{V}_0 \subset V_0$. In practice, $\phi(x)$ is compactly supported and $\phi(x-n)$ is a Resize basis of V_0. Then $\Gamma(\omega) \in \mathcal{A}$ and $\tilde{V}_0 = V_0$.

An open problem. We know that the orthonormal wavelet is a special case of the semi-orthogonal wavelets, and the semi-orthogonal wavelet is also a special case of the biorthogonal wavelets. When we consider the biorthogonal wavelet system, $m_0(\omega)$ and $\tilde{m}_0(\omega)$ must satisfy (2.25), and

$$m_1(\omega) = e^{-i\omega}\overline{\tilde{m}_0(\omega+\pi)}s(2\omega). \tag{2.22}$$

Identities (2.22) and (2.15) lead to

$$\Big(m_0(\omega)\tilde{m}_0(\omega+pi)\Gamma(\omega) - m_0(\omega+pi)\tilde{m}_0(\omega)\Gamma(\omega+\pi)\Big)e^{i\omega}\overline{s(2\omega)} = 0.$$

So, we consider the following equations

$$\begin{cases} \tilde{m}_0(\omega)\overline{m_0(\omega)} + \tilde{m}_0(\omega+\pi)\overline{m_0(\omega+\pi)} = 1 \\ m_0(\omega)\tilde{m}_0(\omega+\pi)\Gamma(\omega) - m_0(\omega+\pi)\tilde{m}_0(\omega)\Gamma(\omega+\pi) = 0 \end{cases}$$

When $m_0(\omega)$ is the characteristic function of some B-spline, we try to find $\tilde{m}_0(\omega)$ with finite coefficients. Unfortunely, $\tilde{m}_0(\omega)$ is always having infinitely many nonzero coefficients, when $m_0(\omega)$ is the characteristic function of some B-spline. If we relax some conditions of $\phi(x)$ and $\tilde{\phi}(x)$, for example, the symmetry or the approximation ability. Then it is hopeful for us to find the semi-orthogonal system $\{\phi(x), \psi(x), \tilde{\phi}(x), \tilde{\psi}(x)\}$ all with compacted support. But we could not do it up to now.

Section 4 Biorthogonal wavelets

Let $\phi(x)$ be defined as in (1.1) and $\tilde{\phi}(x)$ be another scaling function that generats a multiresolution analysis $(\tilde{V}_j)$. We say $\tilde{\phi}(x)$ is dual to $\phi(x)$ (or equivalently $\tilde{\phi}(x-n)$ is dual to $\phi(x-n)$) if

$$\langle \tilde{\phi}(x-n), \phi(x-m) \rangle = \delta_{n,m}. \tag{2.23}$$

Hence for all $n \in \mathbb{Z}$,

$$\begin{aligned} \delta_{0,n} &= \langle \tilde{\phi}(x), \phi(x-n) \rangle \\ &= \int \hat{\tilde{\phi}}(\omega)\overline{\hat{\phi}(\omega)}e^{i\omega n}\, d\omega \\ &= \int_0^{2\pi} e^{i\omega n} \sum_k \hat{\tilde{\phi}}(\omega+2\pi n)\overline{\hat{\phi}(\omega+2\pi n)}\, d\omega. \end{aligned}$$

Since n is arbitrary, we have

$$\sum_k \hat{\tilde{\phi}}(\omega + 2\pi n)\overline{\hat{\phi}(\omega + 2\pi n)} = (2\pi)^{-1} \quad \text{a.e.} \tag{2.24.}$$

Substituting (1.8) leads to

$$\sum_k \tilde{m}_0(\omega + \pi k)\overline{m_0(\omega + \pi k)}\tilde{\phi}(\omega + \pi k)\overline{\phi(\omega + \pi k)} = (2\pi)^{-1} \quad \text{a.e..}$$

Splitting the sum into even and odd k, using the periodicity of $m_0(\omega)$ and $\tilde{m}_0(\omega)$ and applying (2.24), we have

$$\tilde{m}_0(\omega)\overline{m_0(\omega)} + \tilde{m}_0(\omega + \pi)\overline{m_0(\omega + \pi)} = 1 \quad \text{a.e..} \tag{2.25}$$

Similarly, by (1.4) and (1.5) we have the following equations

$$\begin{aligned}
\tilde{m}_0(\omega)\overline{m_1(\omega)} + \tilde{m}_0(\omega + \pi)\overline{m_1(\omega + \pi)} &= 0 \\
\tilde{m}_1(\omega)\overline{m_0(\omega)} + \tilde{m}_1(\omega + \pi)\overline{m_0(\omega + \pi)} &= 0 \\
\tilde{m}_1(\omega)\overline{m_1(\omega)} + \tilde{m}_1(\omega + \pi)\overline{m_1(\omega + \pi)} &= 1 \quad \text{a.e..}
\end{aligned}$$

This is the necessary condition for the biorthogonality. We conclude the above result by

$$\widetilde{M}(\omega)\overline{M^t(\omega)} = 1 \quad \text{a.e.,} \tag{2.26}$$

where

$$M(\omega) = \begin{bmatrix} m_0(\omega) & m_0(\omega + \pi) \\ m_1(\omega) & m_1(\omega + \pi) \end{bmatrix}, \quad \widetilde{M}(\omega) = \begin{bmatrix} \tilde{m}_0(\omega) & \tilde{m}_0(\omega + \pi) \\ \tilde{m}_1(\omega) & \tilde{m}_1(\omega + \pi) \end{bmatrix}.$$

and similarly for $\widetilde{M}(\omega)$. There are several ways to derive $\{\psi, \tilde{\psi}\}$ from $\{\phi, \tilde{\phi}\}$. The following theorem [2] shows a complete characterization of all choices.

Theorem 2.2. *Let ϕ and $\tilde{\phi}$ be biorthogonal scaling functions, and $\{m_0, \tilde{m}_0\}$ be the corresponding 2π-periodic functions. Then $\{m_1, \tilde{m}_1\}$ satisfy* (2.26) *if and only if they are chosen from the class*

$$\tilde{m}_1(\omega) = e^{-i\omega}\overline{m_0(\omega + \pi)s(2\omega)} \quad \text{and} \quad m_1(\omega) = e^{-i\omega}\overline{\tilde{m}_0(\omega + \pi)}s^{-1}(2\omega),$$

for some $s \in \mathcal{A}$.

Usually, we choose $s(\omega) \equiv 1$, that is

$$\widetilde{m}_1(\omega) = e^{-i\omega}\overline{m_0(\omega+\pi)} \quad \text{and} \quad m_1(\omega) = e^{-i\omega}\overline{\widetilde{m}_0(\omega+\pi)}.$$

The wavelets coefficients (d_n) and $(\tilde{d}_n)$ hence become

$$d_n = (-1)^n \tilde{c}_{1-n} \quad \text{and} \quad \tilde{d}_n = (-1)^n c_{1-n}.$$

Here $\tilde{c}_n$ are the coefficients in $\tilde{m}_0(\omega) = \frac{1}{2}\sum \tilde{c}_n e^{-i\omega n}$.

Section 5 The lifting scheme

Let $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ be a biorthogonal system. Now we rewrite (2.25) a little bit more precisely:

$$m_0^{\tilde{\phi}}(\omega)\overline{m_0^{\phi}(\omega)} + m_0^{\tilde{\phi}}(\omega+\pi)\overline{m_0^{\phi}(\omega+\pi)} = 1 \quad \text{a.e.}. \tag{2.27}$$

Let

$$m_0^{\tilde{\phi}_1}(\omega) = m_0^{\tilde{\phi}}(\omega) + e^{-i\omega}\overline{m_0^{\phi}(\omega+\pi)s(2\omega)},$$

where $s(\omega)$ is a trigonometric polynomial in $\mathcal{A}$. Then the corresponding scaling function $\tilde{\phi}_1$ is also a dual scaling function of ϕ [8]. It is easy to chech that $m_0^{\tilde{\phi}_1}(\omega)$ satisfies (2.27), that is, $\langle \phi(x), \tilde{\phi}_1(x-n)\rangle = \delta_{0,n}$.

By Theorem 2.2, m_1^{ϕ} can be chosen with a factor $s(2\omega)$ from $\mathcal{A}$, and since the Wiener class is an algebra, the trigonometric polynomial $m_0^{\tilde{\phi}_1}$ can be written of the form

$$m_0^{\tilde{\phi}_1}(\omega) = m_0^{\tilde{\phi}}(\omega) + m_1^{\tilde{\phi}}(\omega)\overline{s(2\omega)},$$

and

$$m_1^{\phi_1}(\omega) = m_1^{\phi}(\omega) - m_0^{\phi}(\omega)s(2\omega).$$

In this way, the four characteristic trigonometric polynomials $m_0^{\phi}(\omega)$, $m_1^{\phi}(\omega)$, $m_0^{\tilde{\phi}}(\omega)$ and $m_1^{\tilde{\phi}}(\omega)$ change to $m_0^{\phi}(\omega)$, $m_1^{\phi_1}(\omega)$, $m_0^{\tilde{\phi}_1}(\omega)$ and $m_1^{\tilde{\phi}}(\omega)$. Correspondingly, the biorthogonal system $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ changes to $\{\phi, \psi_1, \tilde{\phi}_1, \tilde{\psi}_1\}$. Note that, (1) since ψ_1 is still in the space V_1, its regularity remains the same; (2) the wavelet coefficients of $\tilde{\psi}_1$ is still the same as that of $\tilde{\psi}$, since it only depends on (c_n), which are unchanged.

With proper choices of $s(\omega)$, ψ_1 may have more vanishing moments, and $\tilde{\phi}_1$, $\tilde{\psi}_1$ may have a higher regularity. This is called the *lifting scheme* for biorthogonal wavelets. For instance, by applying the lifting scheme, we can construct the $(N, \tilde{N}) = (1, 3)$ biorthogonal wavelets of [3] from that of $(N, \tilde{N}) = (1, 1)$. See [8] for details. In summary, we have the following theorem.

Theorem 2.3. (Lifting Scheme). *Let* $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ *be an initial set of biorthogonal system. Then a new set* $\{\phi, \tilde{\phi}_1, \psi_1, \tilde{\psi}_1\}$*, which is formally biorthogonal, can be constructed by*

$$\begin{aligned}
\psi_1(x) &= \psi(x) - \sum s_n \phi(x - n) \\
\tilde{\phi}_1(x) &= \sum \tilde{c}_n \tilde{\phi}_1(2x - n) + \sum s_{-n} \tilde{\psi}_1(x - n) \\
\tilde{\psi}_1(x) &= \sum d_n \tilde{\phi}_1(2x - n).
\end{aligned}$$

Here $(\tilde{c}_n)$ *are the scaling coefficients of* $\tilde{\phi}$, (d_n) *are the wavelets coefficients of* $\tilde{\psi}$ *and* (s_n) *is the coefficient of* $e^{-in\omega}$ *in the trigonometric polynomial* $s(\omega)$.

3 The Elevation Scheme

Throughout this chapter, let $\phi(x)$ be the compactly supported orthonormal scaling function defined by Daubechies [4]. It satisfies the scaling equation (1.1) with finitely many nonzero scaling coefficients c_0, c_1, …, c_{2p-1}. The constant p is designated as the *approximation order* of the wavelet $\psi(x)$. That is, $\phi(x-n)$ can span polynomials of degree up to $p-1$ in any finite interval. Let $\psi(x)$ be the compact support orthonormal wavelet associated with $\phi(x)$, then

$$\langle x^m, \psi(x) \rangle = 0, \quad 0 \le m \le p-1.$$

We also review that, by the orthonormality, the scaling coefficients $(c_n)_{n=0}^{2p-1}$ must satisfy the following conditions.

$$\begin{aligned} &(\mathrm{S}) \sum_{n=0}^{2p-1} c_n = 2, \\ &(\mathrm{A}) \sum_{n=0}^{2p-1} (-1)^n n^m c_n = 0 \text{ where } m = 0, \ldots, p-1 \quad (0^0 := 1), \\ &(\mathrm{O}) \sum_{n=0}^{2p-1} c_n c_{n-2m} = \delta_{0m}, \text{ for all } m \in \mathbb{Z}. \end{aligned} \tag{3.1}$$

In this case, $\operatorname{supp} \phi(x) = [0, 2p-1]$ and $d_n = (-1)^n c_{1-n}$, hence $\operatorname{supp} \psi(x) = [1-p, p]$. Following up the idea in [10], define

$$\Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\, dt. \tag{3.2}$$

We call $\Phi(x)$ the *elevation* of $\phi(x)$. We will show that $\Phi(x)$ is still a scaling function that generates a multiresolution analysis. Therefore, there is a wavelet $\Psi(x)$ associated with $\Phi(x)$. Neither $\Phi(x)$ nor $\Psi(x)$ will be orthonormal. Furthermore, the derivatives

$$\Psi'(x-n) = 4\psi(x-n)$$

are orthogonal to each other. However, we will construct a pair of functions $\{\tilde{\Phi}(x), \tilde{\Psi}(x)\}$ such that $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$ is a system of biorthogonal wavelets. With the biorthogonal structure, we will have fast algorithms to change the bases between V_{j+1} and $V_j + W_j$.

Section 1 $\Phi(x)$ is a scaling function

By the definition of $\Phi(x)$, we have

$$\begin{aligned}
\Phi(x) &= \int_{-\infty}^{x} \phi(t) - \phi(t-1)\,dt \\
&= \int_{-\infty}^{x} [\sum_{n=0}^{2p-1} c_n \phi(2t-n)] - [\sum_{n=0}^{2p-1} c_n \phi(2t-n-2)]\,dt \\
&= \int_{-\infty}^{x} \sum_{n=0}^{2p} \frac{(c_{n-1}+c_n)}{2} [\phi(2t-n-2) - \phi(2t-n-1)]\,dt \\
&:= \sum_{n=0}^{2p} h_n \Phi(2x-n).
\end{aligned}$$

Here

$$h_n = \frac{(c_{n-1}+c_n)}{2}, \tag{3.3}$$

note that $c_{-1} = c_{2p} = 0$. Consequently, $\Phi(x)$ satisfies a finite two-scale relation with the scaling coefficients $(h_n)_{n=0}^{2p}$. Hence $\operatorname{supp}\Phi(x) = [0, 2p]$. Furthermore, it is easy to see that

$$\sum_{n=0}^{2p} h_n = \sum_{n=0}^{2p-1} c_n = 2, \tag{3.4}$$

and

$$\sum_{n=0}^{2p} (-1)^n h_n = 0.$$

Proposition 3.1. *If* $\phi(x)$ *is a scaling function that satisfies* (1.1) *and* $\Phi(x)$ *is defined as in* (3.2). *Then*

$$\Phi(x) = \chi_{[0,1]} * \phi(x). \tag{3.5}$$

Furthermore, $\Phi(x)$ is a scaling function with scaling coefficients

$$(h_n) = (\frac{1}{2}, \frac{1}{2}) * (c_n).$$

Proof: By (3.2) and (1.6),

$$\begin{aligned}
\Phi(x) &= \int_{-\infty}^{x} \phi(t) - \phi(t-1)\,dt \\
&= \int_{-\infty}^{x} \phi(t)\,dt - \int_{-\infty}^{x} \phi(t-1)\,dt \\
&= \int_{-\infty}^{x} \phi(t)\,dt - \int_{-\infty}^{x-1} \phi(t)\,dt \\
&= \int_{x-1}^{x} \phi(t)\,dt \\
&= \int_{-\infty}^{\infty} \chi_{[0,1]}(x-t)\phi(t)\,dt \\
&= \chi_{[0,1]} * \phi(x).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\hat{\Phi}(\omega) &= \widehat{\chi_{[0,1]}}(\omega) \cdot \hat{\phi}(\omega) \\
&= \frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} e^{-i\frac{\omega}{2}}\, \hat{\phi}(\omega) \\
&= (\frac{1}{2} + \frac{1}{2}e^{-i\frac{\omega}{2}})\widehat{\chi_{[0,1]}}(\frac{\omega}{2})\, m_0^{\phi}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}) \\
&:= m_0^{\Phi}(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2}).
\end{aligned}$$

Here

$$\begin{aligned}
m_0^{\Phi}(\omega) &= (\frac{1}{2} + \frac{1}{2}e^{-i\omega})m_0^{\phi}(\omega) \\
&= \frac{1}{2}(\frac{1}{2} + \frac{1}{2}e^{-i\omega})\sum_n c_n e^{-in\omega} \\
&:= \frac{1}{2}\sum_n h_n e^{-in\omega}.
\end{aligned}$$

Note that the coefficients (h_n) can be obtained by the vector convolution of $(\frac{1}{2}, \frac{1}{2})$ and (c_n), see (1.7). Take the inverse Fourier transform on both sides of

$$\hat{\Phi}(\omega) = m_0^{\Phi}(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2}),$$

we have

$$\Phi(x) = \sum_n h_n \Phi(2x - n).$$

Hence $\Phi(x)$ satisfies a scaling equation. □

Remark. Note that $\widehat{\chi_{[0,1]}}(\omega)$ is continuous at $\omega = 0$. Since $\hat{\phi}(\omega)$ is also a continuous function, so $\Phi(\omega)$ is continuous. Thus, we have $\hat{\Phi}(0) = 1$ and $\hat{\Phi}(2n\pi) = 0$ for $n \neq 0$.

Section 2 $\Phi(x)$ generates a multiresolution analysis

Among the properties of (c_n) listed in (3.1), (S) follows from (1.1), (A) is necessary and sufficient for the degree of approximation, and (O) is necessary for the orthonormality. In (3.4), we have seen that (h_n) satisfies (S). Now we will show that (h_n) will satisfy (A) with one degree higher than that of (c_n), but (h_n) will not satisfy (O).

Proposition 3.2. *If* $\phi(x)$ *is a scaling function that satisfies* (1.1) *and* (c_n) *satisfies the condition* (A) *in* (3.1), *and if* $\Phi(x)$ *is defined as in* (3.2). *Than the scaling coefficients* (h_n) *of* $\Phi(x)$ *satisfy*

$$\sum_{n=0}^{2p}(-1)^n n^m h_n = 0, \text{ where } m = 0, \dots, p \text{ (with } 0^0 := 1). \tag{3.6}$$

Proof: For $m = 0$,

$$\begin{aligned}\sum_{n=0}^{2p}(-1)^n h_n &= \sum_{n=0}^{2p}(-1)^n\left(\frac{c_n + c_{n-1}}{2}\right)\\ &= \frac{1}{2}\left[\sum_{n=0}^{2p-1}(-1)^n c_n + \sum_{n=0}^{2p-1}(-1)^{n+1} c_n\right]\\ &= \frac{1}{2}\left[\sum_{n=0}^{2p-1}(-1)^n c_n - \sum_{n=0}^{2p-1}(-1)^n c_n\right]\\ &= 0.\end{aligned}$$

Assume that (3.6) holds for $m = 0, \ldots, k-1$, where $0 < k \le p$. Then

$$\begin{aligned}
\sum_{n=0}^{2p}(-1)^n n^k h_n &= \sum_{n=0}^{2p}(-1)^n n^k (\frac{c_n + c_{n-1}}{2}) \\
&= \frac{1}{2}\sum_{n=0}^{2p}(-1)^n n^k c_n + \frac{1}{2}\sum_{n=0}^{2p}(-1)^n n^k c_{n-1} \\
&= \frac{1}{2}\sum_{n=0}^{2p-1}(-1)^n n^k c_n + \frac{1}{2}\sum_{n=1}^{2p}(-1)^n n^k c_{n-1} \\
&= \frac{1}{2}[\sum_{n=0}^{2p-1}(-1)^n n^k c_n + \sum_{n=0}^{2p-1}(-1)^{n+1}(n+1)^k c_n] \\
&= \frac{1}{2}\sum_{n=0}^{2p-1}(-1)^n \left[n^k - (n+1)^k\right] c_n \\
&= \frac{1}{2}\sum_{n=0}^{2p-1}(-1)^{n+1}[\sum_{\ell=0}^{k-1}\binom{k}{\ell} n^\ell] c_n \\
&= -\frac{1}{2}\sum_{\ell=0}^{k-1}\binom{k}{\ell}[\sum_{n=0}^{2p-1}(-1)^n n^\ell c_n] \\
&= 0.
\end{aligned}$$

Hence, by induction, we have proved (3.6). □

The above proposition means that polynomials up to degree p can be a linear combination of the translations of $\Phi(x)$, if provided $\sum \Phi(x-n) \equiv 1$. It is well known that the $\phi(x-n)$ of [4] can express the constant functions, namely $\sum \phi(x-n) \equiv 1$. Although $\Phi(x-n)$ is not an orthonormal set (we will show that in the next section), Φ still preserves this property of ϕ such that $\sum \Phi(x-n) \equiv 1$.

Proposition 3.3. *If* $\phi(x)$ *is a scaling function such that*

$$\sum_{n\in\mathbb{Z}} \phi(x-n) = 1, \quad \forall x \in \mathbb{R},$$

and $\Phi(x)$ *is defined as in* (3.2). *Then*

$$\sum_{n\in\mathbb{Z}} \Phi(x-n) = 1, \quad \forall x \in \mathbb{R}.$$

Proof: For all $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \Phi(x-n) &= \sum_{n\in\mathbb{Z}} \phi * \chi_{[0,1]}(x-n) \\
&= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \phi(x-n-t)\chi_{[0,1]}(t)\,dt \\
&= \int_{-\infty}^{\infty} \Big(\sum_{n\in\mathbb{Z}} \phi(x-n-t)\Big)\chi_{[0,1]}(t)\,dt \\
&= \int_{-\infty}^{\infty} \chi_{[0,1]}(t)\,dt \\
&= 1.
\end{aligned}$$

□

Remark. By the fact that $\phi(x-n)$ can span the polynomials up to degree $p-1$ and the above proposition, we can prove that $\Phi(x-n)$ can span the polynomials up to degree p. Specifically,

$$\sum_{n\in\mathbb{Z}} n^m \Phi(x-n)$$

is a polynomial of degree m, for $0 \le m \le p \quad (0^0 := 1)$.

Let $V_0 = \text{span}\,\{\Phi(x-n)\}$, we would like to check if $\Phi(x-n)$ forms a Riesz basis of V_0. It amounts to show that there are constants $0 < A \le B$ with

$$A \le \sum_{n=-\infty}^{\infty} |\hat{\Phi}(\omega + 2\pi n)|^2 \le B, \quad \forall \omega \in \mathbb{R}.$$

This is a well-known property, see, for example, [2]. If $\Phi(x-n)$ forms a Riesz basis of V_0 and by $\hat{\Phi}(0) = 1$, then $\Phi(x)$ generates a multiresolution analysis of $L^2(\mathbb{R})$.

Theorem 3.1. *If ϕ is a scaling function that generates a multiresolution anaysis and $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the function $\Phi = \phi * \chi_{[0,1]}$ also generates a multiresolution analysis.*

When we prove this proposition, we need the following theorem and lemmas.

Theorem 3.2. *Let* $f \in L^1(\mathbb{R})$. *Then its Fourier transform* $\hat{f}$ *satisfies:*

(1) $\hat{f} \in L^\infty(\mathbb{R})$ *with* $\|\hat{f}\|_\infty \le \|f\|_1$;

(2) $\hat{f}$ *is uniformly continuous on* $\mathbb{R}$;

(3) *if the derivative* f' *of* f *also exists and is in* $L^1(\mathbb{R})$, *then*

$$\widehat{f'}(\omega) = i\omega\hat{f}(\omega); \text{ and } \hat{f}(\omega) \to 0, \text{ as } |\omega| \to \infty.$$

Proof: Assertion (1) is straight forward. For all $\omega \in \mathbb{R}$

$$\begin{aligned} |\hat{f}(\omega)| = |\int e^{-i\omega x} f(x)\,dx| &\le \int |e^{-i\omega x} f(x)|\,dx \\ &\le \|e^{-i\omega x}\|_\infty\,\|f\|_1 \le \|f\|_1. \end{aligned}$$

To prove (2), let δ be chosen arbitrarily and consider

$$\begin{aligned} \sup_\omega |\hat{f}(\omega+\delta) - \hat{f}(\omega)| &= \sup_\omega |\int e^{-i\omega x}(e^{-i\delta x} - 1) f(x)\,dx| \\ &\le \int |e^{-i\delta x} - 1|\,|f(x)|\,dx. \end{aligned}$$

Now, since $|e^{-i\delta x} - 1|\,|f(x)| \le 2|f(x)| \in L^1(\mathbb{R})$ and $|e^{-i\delta x} - 1| \to 0$ as $\delta \to 0$. By the Lebesgue Dominated Convergence Theorem, $\sup_\omega |\hat{f}(\omega+\delta) - \hat{f}(\omega)| \to 0$ as $\delta \to 0$. So $\hat{f}$ is uniformly continuous on $\mathbb{R}$.

To prove (3), we apply the theorem of the integration by parts:

$$\begin{aligned} \widehat{f'}(\omega) &= \int e^{-i\omega x} f'(x)\,dx = \int e^{-i\omega x}\,df(x) \\ &= e^{-i\omega x}\,f(x)\big|_{-\infty}^{\infty} - \int f(x)\,de^{-i\omega x} = i\omega \int e^{-i\omega x} f(x)\,dx \\ &= i\omega\hat{f}(\omega), \end{aligned}$$

Because $f \in L^1(\mathbb{R})$, $f(x) \to 0$ as $|x| \to \infty$. If f' is in $L^1(\mathbb{R})$, then by the proof of the assertion (1)

$$|\hat{f}(\omega)| = \frac{1}{|\omega|}|\widehat{f'}(\omega)| \le \frac{1}{|\omega|}\|f'\|_1 \to 0,$$

as $|\omega| \to \infty$. □

Lemma 3.1. *Let* $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and* $\Phi = \chi_{[0,1]} * \phi$. *Then the function*

$$\Gamma(\omega) := \sum_{n=-\infty}^{\infty} |\hat{\Phi}(\omega + 2\pi n)|^2, \quad \textit{for all } \omega \in \mathbb{R} \tag{3.7}$$

is a continuous function of ω *on* $(-2\pi, 2\pi)$.

Proof: By the previous theorem, $\hat{\phi}(\omega)$ is a continuous function in $\mathbb{R}$. Then

$$\hat{\Phi}(\omega + 2\pi n) = \left(\frac{\sin \frac{\omega+2\pi n}{2}}{\frac{\omega+2\pi n}{2}} \right) e^{-i\frac{\omega+2\pi n}{2}} \hat{\phi}(\omega + 2\pi n) \tag{3.8}$$

is also continuous on $\mathbb{R}$. Define a sequence of functions (Γ_k) by

$$\Gamma_k(\omega) = \sum_{n=-k}^{n=k} |\hat{\Phi}(\omega + 2\pi n)|^2 = \sum_{n=-k}^{n=k} \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega+2\pi n}{2}} \right)^2 |\hat{\phi}(\omega + 2\pi n)|^2.$$

Γ_k is a finite sum of the continuous functions, then Γ_k is continuous, too. To show $\Gamma(\omega)$ is continuous, it is sufficient to show that $\Gamma_k \to \Gamma$ uniformly on $(-2\pi, 2\pi)$.

For all $k \geq 2$,

$$\begin{aligned}
|\Gamma_k - \Gamma| &= \left| \sum_{n=-\infty}^{-k-1} \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega+2\pi n}{2}} \right)^2 |\hat{\phi}(\omega + 2\pi n)|^2 + \sum_{n=k+1}^{\infty} \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega+2\pi n}{2}} \right)^2 |\hat{\phi}(\omega + 2\pi n)|^2 \right| \\
&\leq \sum_{n=-\infty}^{-k-1} \frac{4}{|\omega + 2\pi n|^2} \|\phi\|_1^2 + \sum_{n=k+1}^{\infty} \frac{4}{|\omega + 2\pi n|^2} \|\phi\|_1^2 \\
&\leq \frac{2}{\pi^2} \|\phi\|_1^2 \left(\sum_{n=k-1}^{\infty} \frac{1}{n^2} \right).
\end{aligned}$$

So $\forall\, \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $\forall k \geq N$, we have $\|\Gamma_k - \Gamma\|_\infty < \varepsilon$, and $\Gamma_k \to \Gamma$ uniformly on $(-2\pi, 2\pi)$. ☐

Lemma 3.2. *If the function* $\Gamma(\omega)$ *is defined as* (3.7), *then* $\Gamma(\omega) > 0$ *for all* $\omega \in [0, 2\pi)$.

Proof: if $\exists\, \omega_0 \in [0, 2\pi)$ such that $\Gamma(\omega_0) = 0$, then

$$\sum_{n=-\infty}^{\infty} \left(\frac{\sin \frac{\omega_0}{2}}{\frac{\omega_0+2\pi n}{2}} \right)^2 |\hat{\phi}(\omega_0 + 2\pi n)|^2 = 0.$$

We have $\sin \frac{\omega_0}{2} = 0$ or $\hat{\phi}(\omega_0 + 2\pi n) = 0$, for all $n \in \mathbb{Z}$. If $\sin \frac{\omega_0}{2} = 0$, then $\omega_0 = 0$. And by (3.8), we have

$$\hat{\Phi}(0) = 1 \text{ and } \hat{\Phi}(2\pi n) = 0 \quad \text{for } n \neq 0,$$

then $\Gamma(0) = 1$. So $\Gamma(\omega_0) = 0$ must imply that $\hat{\phi}(\omega_0 + 2\pi n) = 0, \quad \forall n \in \mathbb{Z}$. This contradicts the assumption $0 < A_\phi \leq \sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega_0 + 2\pi n)|^2$. So $\Gamma(\omega) > 0$, $\forall \omega \in [0,2\pi)$. □

Now, we turn to prove the Theorem 3.1:

Proof: The inequality $\sum_{n=-\infty}^{\infty} |\hat{\Phi}(\omega + 2\pi n)|^2 \leq B$ is easy! Because $\phi(x-n)$ is a Riesz basis. Hence $\exists\ 0 < A_\phi \leq B_\phi$ such that

$$A_\phi \leq \sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi n)|^2 \leq B_\phi .$$

Since

$$|\hat{\Phi}(\omega + 2\pi n)|^2 = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega + 2\pi n}{2}} \right)^2 |\hat{\phi}(\omega + 2\pi n)|^2 \leq |\hat{\phi}(\omega + 2\pi n)|^2$$

for all $\omega \in \mathbb{R}$ and for all $n \in \mathbb{Z}$. We have

$$\sum_{n=-\infty}^{\infty} |\hat{\Phi}(\omega + 2\pi n)|^2 \leq \sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi n)|^2 \leq B_\phi .$$

On the other hand, since Γ is continuous on $(-2\pi, 2\pi)$ and Γ is 2π-periodic, then $\Gamma(\omega) > 0$ on $[0,2\pi]$ by Lemma. Hence $\exists\ A_\Phi > 0$ such that $A_\Phi \leq \Gamma(\omega), \quad \forall \omega \in \mathbb{R}$. We have proved that $\Phi(x-n)$ forms a Riesz basis. □

Since $\Phi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\hat{\Phi}(0) = 1$ and $\Phi(x-n)$ form a Riesz basis. $\Phi(x)$ indeed generates a new multiresolution analysis (V_j) of $L^2(\mathbb{R})$.

Section 3 $\Psi'(x-n)$ are orthogonal

We have seen that $\Phi(x)$ generates a multiresolution analysis. At this point, we do not know whether $\Phi(x-n)$ are orthogonal or not. However, by the numerical tests on several basic cases, we believe they are not orthogonal. Therefore, we consider the general setting of the biorthogonal system, see (2.25). We will derive the associated wavelet $\Psi(x)$ from the dual of $\Phi(x)$. The new biorthogonal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$ must satisfy (2.26). Because

$$m_0^\Phi(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right) m_0^\phi(\omega), \tag{3.9}$$

we chose

$$m_0^{\tilde{\Phi}}(\omega) = \begin{cases} \dfrac{2e^{-i\omega}}{1+e^{-i\omega}} m_0^{\phi}(\omega) & \text{if } \omega \neq (2n+1)\pi, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$m_1^{\tilde{\Phi}}(\omega) = e^{-i\frac{\omega}{2}}\overline{m_0^{\Phi}(\omega+\pi)}, \qquad m_1^{\Phi}(\omega) = e^{-i\frac{\omega}{2}}\overline{m_0^{\tilde{\Phi}}(\omega+\pi)}.$$

With this choices of $m_1^{\Phi}(\omega)$ and $m_1^{\tilde{\Phi}}(\omega)$, the wavelets coefficients (g_n) and $(\tilde{g}_n)$ are

$$g_n = (-1)^n \tilde{h}_{1-n}, \quad \tilde{g}_n = (-1)^n h_n 1 - n. \tag{3.10}$$

Then, see Section 4 of Chapter 2, we can construct the biorthogonal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$. For this chosen, we will see that (1) the relation between $\tilde{\Phi}$ and $\tilde{\phi}$ is "reciprocal" to the relation between Φ and ϕ; (2) $\Psi'(x-n)$ are orthogonal to each other and to $\Phi'(x-n)$; (3) $\Phi(x-n)$ are *not* orthogonal.

Proposition 3.4. *Let ϕ be an orthonormal scaling function and let Φ be defined by* (3.5). *If $\tilde{\Phi}$ is defined by*

$$\chi_{[-1,0]} * \tilde{\Phi} = \phi. \tag{3.11}$$

Then $\tilde{\Phi}$ is a biorthogonal dual of Φ.

Proof: If there is a function $\tilde{\Phi}(x)$ such that $\chi_{[-1,0]} * \tilde{\Phi} = \phi$. We have

$$\widehat{\chi_{[-1,0]}}(\omega)\hat{\tilde{\Phi}}(\omega) = \hat{\phi}(\omega),$$

that is

$$\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)e^{i\frac{\omega}{2}}\hat{\tilde{\Phi}}(\omega) = \hat{\phi}(\omega).$$

Then

$$m_0^{\tilde{\Phi}}(\omega) = \frac{2}{1+e^{i\omega}} m_0^{\phi}(\omega) \quad \text{a.e..} \tag{3.12}$$

Surely, $m_0^{\Phi}(\omega)$ and $m_0^{\tilde{\Phi}}(\omega)$ satisfy (2.25), and $\tilde{\Phi}(x)$ is a dual to $\Phi(x)$. □

Remark. In practice, the function $m_0^{\tilde{\phi}}(\omega)$ will have a factor of $(1+e^{-i\omega})/2$. Therefore, in practice, $m_0^{\tilde{\Phi}}(\omega)$ has finitely many coefficients; in other words, $\tilde{\Phi}(x)$ is compactly supported. However, we can see from (3.11) that the regularity of $\tilde{\Phi}(x)$ will be less than that of $\tilde{\phi}(x)$. We have seen this phenomenon in the biorthogonal wavelets derived by [3].

We have no regard for whether the pair $\{\tilde{\Phi}, \tilde{\Psi}\}$ exist in $L^2(\mathbb{R})$. As long as the scaling coefficients $(\tilde{c}_n)$ is a finite sequence, by Theorem 2.1, $\tilde{\Phi}$ and $\tilde{\Psi}$ are distributions at least. We know that $\Phi(x) = \int_{-\infty}^{x} \phi(t) - \phi(t-1)\,dt$ if and only if $\Phi = \phi * \chi_{[0,1]}$. It is easy to see that $\chi_{[-1,0]} * \tilde{\Phi} = \tilde{\phi}$ if and only if $\tilde{\phi}'(x) = \tilde{\Phi}(x+1) - \tilde{\Phi}(x)$. Then it is easy to evaluate the function $\tilde{\Phi}(x)$, provided that $\tilde{\Phi}$ has compact support and the value of $\tilde{\phi}'(x)$ is known.

The following proposition shows the beautiful relation between the elevated wavelet Ψ and the original wavelet ψ. That is

$$\Psi(x) = 4\int_{-\infty}^{x} \psi(t)\,dt \quad \forall x \in \mathbb{R}. \tag{3.13}$$

Proposition 3.5. *Let* $\Phi(x)$ *be the elevation of the orthonormal scaling function* $\phi(x)$ *and* $\tilde{\Phi}(x)$ *is defined by* (3.11). *If we construct the wavelet* $\Psi(x)$ *from* $m_1^{\Phi}(\omega)$, *then* $\Psi'(x-n)$ *is an orthogonal set.*

Proof: Since $\Phi(x) = \chi_{[0,1]} * \phi(x)$ and $\phi(x) = \chi_{[-1,0]} * \tilde{\Phi}(x)$, we have $m_0^{\Phi}(\omega)$ satisfies (3.9) and $m_0^{\tilde{\Phi}}(\omega)$ satisfies (3.12). We have

$$\begin{aligned} m_1^{\Phi}(\omega) &= e^{-i\frac{\omega}{2}}\left(\frac{1+e^{-i(\omega+\pi)}}{2}\right)^{-1}\overline{m_0(\omega+\pi)} \\ &= m_1(\omega)\left(\frac{1+e^{-i(\omega+\pi)}}{2}\right)^{-1}. \end{aligned}$$

By Theorem 3.2, $\widehat{\Psi'}(\omega) = i\omega\hat{\Psi}(\omega)$, and we have

$$\begin{aligned} i\omega\hat{\Psi}(\omega) &= i\omega\, m_1(\frac{\omega}{2})\left(\frac{1+e^{-i(\frac{\omega}{2}+\pi)}}{2}\right)^{-1}\hat{\Phi}(\frac{\omega}{2}) \\ &= i\omega\, m_1(\frac{\omega}{2})\left(\frac{2}{1-e^{-i\frac{\omega}{2}}}\right)\frac{2(1-e^{-i\frac{\omega}{2}})}{i\omega}\hat{\phi}(\frac{\omega}{2}) \\ &= 4\, m_1(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}) \\ &= 4\,\hat{\psi}(\omega) \quad \text{a.e..} \end{aligned}$$

Take the inverse Fourier transform on both sides of the above equation, we get

$$\Psi'(x) = 4\psi(x), \quad \forall x \in \mathbb{R}.$$

□

Remark. Let $\phi(x)$ be the Daubechies orthonormal scaling function and $\phi(x) \in \mathcal{C}\surd$ for some p. If we define $\phi_r(x)$ by

$$\phi_r(x) = \phi * \chi_{[0,1]} * \ldots * \chi_{[0,1]} \quad \text{(r times)},$$

for $1 \leq r < p$. Then, if $\{\phi_r, \psi_r, \tilde{\phi}_r, \tilde{\psi}_r\}$ is the biorthogonal system as chosen in this section, we have

$$\psi_r^{(r)}(x) = 4^r \psi(x).$$

That is, $\psi_r^{(r)}(x-n)$ is an orthogonormal set.

Now we are at the position to prove that, under this setting, $\Phi(x-n)$ are not orthogonal. First, we introduce the following lemma [11]. (For the readers convenience, I include his proof here.)

Lemma 3.3. *Let $\phi(x)$ be an orthonormal scaling function and $\psi(x)$ be the associated orthonormal wavelet. If both ϕ and ψ are in $\mathcal{C}^p$ for some $p > 0$, then*

$$\langle \psi^{(r)}(x), \psi^{(s)}(x-n) \rangle = 2^{r+s} \langle \phi^{(r)}(x), \phi^{(s)}(x-2n) \rangle - \langle \phi^{(r)}(x), \phi^{(s)}(x-n) \rangle,$$

where $1 \leq r, s \leq p$.

Proof: By Theorem 3.2 and the Parseval identity, we have

$$\begin{aligned}
\langle \psi^{(r)}(x), \psi^{(s)}(x-n) \rangle &= \int (-i\omega)^r \hat{\psi}(\omega) \overline{e^{-i\omega n} (-i\omega)^s \hat{\psi}(\omega)} \, d\omega \\
&= \int (-1)^{(r-s)} (i\omega)^{r+s} |m_1(\frac{\omega}{2})|^2 e^{i\omega n} |\hat{\phi}(\frac{\omega}{2})|^2 \, d\omega \\
&= \int (-1)^{(r-s)} (i\omega)^{r+s} (1 - |m_0(\frac{\omega}{2})|^2) e^{i\omega n} |\hat{\phi}(\frac{\omega}{2})|^2 \, d\omega \\
&= \int (-1)^{(r-s)} (i\omega)^{r+s} e^{i\omega n} |\hat{\phi}(\frac{\omega}{2})|^2 \, d\omega \\
&\qquad - \int (-1)^{(r-s)} (i\omega)^{r+s} |m_0(\frac{\omega}{2})|^2 e^{i\omega n} |\hat{\phi}(\frac{\omega}{2})|^2 \, d\omega \\
&= 2^{r+s} \langle \phi^{(r)}(x), \phi^{(s)}(x-2n) \rangle - \langle \phi^{(r)}(x), \phi^{(s)}(x-n) \rangle.
\end{aligned}$$

This completes the proof. $\Box$

Proposition 3.6. *Let $\phi(x)$ be an orthonormal scaling function, and $\Phi(x)$ is the elevation of $\phi(x)$. Then $\Phi(x-n)$ is not orthogonal.*

Proof: If $\Phi(x-n)$ were orthogonal, we can modify the functions by a scalar to make them orthonormal. And note that Φ and Ψ are in $\mathcal{C}^1$. Then, apply the previous lemma with $r = s = 1$, we have

$$\begin{aligned}\delta_{0,n} &= \langle \Psi'(x), \Psi'(x-n)\rangle \\ &= 4\langle \Phi'(x), \Phi'(x-2n)\rangle - \langle \Phi'(x), \Phi'(x-n)\rangle \\ &= \langle \Phi'(x), \Phi'(x-n)\rangle.\end{aligned}$$

But it is easy to check that

$$\langle \Phi'(x), \Phi'(x-n)\rangle = \begin{cases} 2 & \text{if } n = 0, \\ -1 & \text{if } n = 1 \text{ or } -1, \\ 0 & \text{otherwise.} \end{cases}$$

This is a contradiction, so $\Phi(x-n)$ cannot be orthogonal. $\square$

Section 4 The biorthogonal filter coefficients

Let $\phi(x)$ be an orthonormal scaling function, $\psi(x)$ be the associated orthonormal wavelet, and $(c_n)_{n=0}^{N-1}$ be the nonzero scaling coefficients. Suppose $\phi(x)$ has the approximation degree p, then $N = 2p$. Let $\Phi(x)$ and $\tilde{\Phi}(x)$ be the pair of elevated scaling functions defined in the previous section. Let (h_n), $(\tilde{h}_n)$ be the scaling coefficients of $\Phi(x)$ and $\tilde{\Phi}(x)$, respectively. By (3.3), we can easily derive the following algorithm.

Algorithm 1.

```
$h_0 = \frac{c_0}{2}$,  $h_N = \frac{c_{N-1}}{2}$
FOR $n = 1$ TO $N-1$
    $h_n = \frac{c_{n-1} + c_n}{2}$
END
```

For $(\tilde{h}_n)$, we compare the coefficients of $m_0^{\tilde{\Phi}}$ and m_0^{ϕ}. For instance, see (3.12). Here we assume the pratical case such that $m_0^{\phi}(\omega)$ has at least a factor of $(1 + e^{-i\omega})/2$. The following algorithm derives $(\tilde{h}_n)$ from (c_n).

Algorithm 2.

```
$\tilde{h}_{-1} = 0$
FOR $n = 0$ TO $N - 2$
    $\tilde{h}_n = 2c_n - \tilde{h}_{n-1}$
END
```

Once we have the scaling coefficients (h_n) and $(\tilde{h}_n)$, the wavelets coefficients can be obtained by (3.10). By (3.9) we observe that the number of vanishing moments of Ψ is one more than that of ψ. Conversely, by (3.12), we observe that the number of vanishing moments of $\tilde{\Psi}$ is one less than that of ψ. By (3.5), we can also see that Φ has higher regularity than that of ϕ; and by (3.11), we see that $\tilde{\Phi}$ has less regularity than that of ϕ. The scaling coefficients $(\tilde{h}_n)$ of $\tilde{\Phi}$ has the similar property as that for (h_n) stated in Proposition 3.2.

Proposition 3.7. *Let* $\phi(x)$ *be defined as in* (1.1) *and* (c_n) *satisfy the condition* (A) *in* (3.1). *If* $\tilde{\Phi}(x)$ *is defined by* $\chi_{[-1,0]} * \tilde{\Phi} = \phi$, *then the scaling coefficients* $(\tilde{h}_n)$ *of* $\tilde{\Phi}(x)$ *satisfy*

$$\sum_{n=0}^{2p-2} (-1)^n n^m \tilde{h}_n = 0, \text{ where } m = 0, \ldots, p-2 \quad (0^0 := 1).$$

Proof: By Algorithm 2 and the fact that $\sum(-1)^n c_n = 0$, we have

$$\tilde{h}_n = 2 \sum_{k=n+1}^{2p-1} (-1)^{k-n-1} c_k, \quad n = 0, \ldots, 2p-2.$$

By the equation

$$\sum_{n=0}^{2p-1} n^m \sum_{k=n}^{2p-1} (-1)^k c_k = \sum_{n=0}^{2p-1} \left(\sum_{k=0}^{n} k^m \right) (-1)^n c_n,$$

the fact that $\sum_{k=1}^{n} k^m$ is a polynomial in n of degree $m+1$, and the condition (A) for $(\tilde{h}_n)$, we have

$$\sum_{n=1}^{2p-1} n^m \sum_{k=n}^{2p-1} (-1)^k c_k = 0, \text{ for } m = 0, \ldots, p-2.$$

Then, for $0 \le m \le p-2$,

$$\begin{aligned}
\sum_{n=0}^{2p-2}(-1)^n n^m \tilde{h}_n &= 2\sum_{n=0}^{2p-2}(-1)^n n^m \sum_{i=n+1}^{2p-1}(-1)^{i-n-1}c_i \\
&= -2\sum_{n=1}^{2p-1}(n-1)^m \sum_{i=n}^{2p-1}(-1)^i c_i \\
&= -2\sum_{n=1}^{2p-1}\left(\sum_{\ell=0}^{m}\binom{m}{\ell}(-1)^\ell n^\ell\right)\sum_{i=n}^{2p-1}(-1)^i c_i \\
&= -2\sum_{\ell=0}^{m}\binom{m}{\ell}(-1)^\ell\left(\sum_{n=1}^{2p-1} n^\ell \sum_{i=n}^{2p-1}(-1)^i c_i\right) \\
&= 0.
\end{aligned}$$

This completes the proof. $\square$

Section 5 Generalizations

Let us now turn to the general setting and assume that $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ is a biorthogonal system. Even if $\phi(x)$ is not an orthonormal scaling function (and $\psi(x)$ is not an orthonormal wavelet), many properties of $\Phi(x)$ (the elevation of $\phi(x)$) and $\Psi(x)$ can still be preserved. For example, $\Phi(x)$ is still a scaling function and still generates a multiresolution analysis of $L^2(\mathbb{R})$; the approximation degree of $\Phi(x)$ is better than $\phi(x)$, etc. Specifically, all theorems and propositions in Section 1 and 2 of this chapter hold without modifications. In Section 3, ϕ would be changed to $\tilde{\phi}$ in (3.11), $m_0^{\phi}(\omega)$ would be changed to $m_0^{\tilde{\phi}}(\omega)$ in (3.12), and (3.13) would be still true but $\Psi'(x)$ would not be orthogonal. The dual functions $\{\tilde{\Phi}, \tilde{\Psi}\}$ would be constructed in the same way as described in Section 3. All of the proofs remain the same with small modifications. In Section 4, Algorithm 1 would hold without modification, Algorithm 2 and Proposition 3.7 would hold with (c_n) changed to $(\tilde{c}_n)$ and ϕ changed to $\tilde{\phi}$.

From equation (3.12), we see that the regularity of $\tilde{\Phi}$ is worse then that of $\tilde{\phi}$. One advantage of starting the elevation scheme from a biorthogornal system, instead of an orthonormal system, is that we can get a smoother biorthogornal system $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$. If we do not need the property of orthogonality for $\Psi'(x-n)$, we can apply the lifting scheme to (1) first elevate from $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$ to $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$, then lift $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$;

or (2) first lift $\{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$, then elevate the lifted biorthogonal system to $\{\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}\}$. In either way, we can raise regularity of $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$.

4 Examples

Section 1 The spline case

Given an index $N \geq 1$, let $\phi_{(N)}(x)$ be the *B-spline* of order $N-1$; that is, $\phi_{(N)}(x) = B_{N-1}(x)$. Here $B_N(x)$ is defined recursively by

$$B_N(x) = B_{N-1} * B_0 \quad \forall N \geq 1,$$

with $B_0 = \chi_{[0,1]}$. In this notation, $\phi_{(N)}(x)$ do not have the symmetry on the origin. The Fourier transform of $\phi_{(N)}$ is given by

$$\hat{\phi}_{(N)}(\omega) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^N,$$

and

$$m_0^{\phi_{(N)}}(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^N.$$

The set of B-spline functions $\phi_{(N)}(x-k)$ is not orthonormal for $N > 1$, we consider the biorthogornal system $\{\phi_{(N)}, \tilde{\phi}_{(N,\tilde{N})}\}$. A biorthogornal system must satisfies (2.25), and the possible solutions $m_0^{\tilde{\phi}_{(N,\tilde{N})}}(\omega)$ [3] are given by

$$m_0^{\tilde{\phi}_{(N,\tilde{N})}}(\omega) = \cos^{\tilde{N}}\frac{\omega}{2}\left[\sum_{\ell=0}^{k-1}\binom{k-1+\ell}{\ell}\sin^{2\ell}\frac{\omega}{2} + \sin^{2k}\frac{\omega}{2}R(\cos\omega)\right],$$

where $\tilde{N} \leq 1$, $N + \tilde{N} = 2k$ is even, R is an odd polynomail. We write the corresponding scaling function by $\tilde{\phi}_{(N,\tilde{N})}$. Note that $\phi_{(N)}$ is the elevation of $\phi_{(N-1)}$, and $\tilde{\phi}_{(N,\tilde{N})}$ is the elevation of $\tilde{\phi}_{(N-1,\tilde{N}+1)}$.

For instance, in case of the $(N, \tilde{N}) = (1,1)$ biorthogonal system of [3], the graphics $\phi_{(1)}$ and $\psi_{(1)}$ are in Figure 1 and 2, respectively.

After a step of the lifting scheme, we obtain the $(N, \tilde{N}) = (1,3)$ biorthogonal system of [3]. See Figure 3 and 4.

Taking the $(N, \tilde{N}) = (1,3)$ biorthogonal system and apply the elevation scheme, we will arrive the $(N, \tilde{N}) = (2,2)$ biorthogonal system of [3]. See Figure 5 and 6.

Section 2 The orthonormal case

In this case ϕ_p is the Daubechies orthonormal scaling function of order p. The scaling coefficients $(c_n)_{n=0}^{2p-1}$ satisfy (3.1). For example, in case of $p = 1$, we have

$$c_0 = 1 \text{ and } c_1 = 1.$$

This is the Haar orthonormal system (Figure 1). In case of $p = 2$, we have

$$c_0 = \frac{1+\sqrt{3}}{4}, c_1 = \frac{3+\sqrt{3}}{4}, c_2 = \frac{3-\sqrt{3}}{4}, c_3 = \frac{1-\sqrt{3}}{4}.$$

The graphics ϕ_2 and ψ_2 are in Figure 7 and 8.

We write Φ_p the elevation of ϕ_p and $\tilde{\Phi}_p$ the dual of Φ_p with minmal support. The dual $\tilde{\Phi}_p$ belongs to $L^2(\mathbb{R})$ for all $n > 2$.

The scaling coefficients of Φ_2 is

$$h_0 = \frac{1+\sqrt{3}}{8}, h_1 = \frac{2+\sqrt{3}}{4}, h_2 = \frac{3}{4}, h_3 = \frac{2-\sqrt{3}}{4}, h_4 = \frac{1-\sqrt{3}}{8}.$$

And the graphics of Φ_2 is in Figure 9, Ψ_2 is in Figure 10. Although, $\tilde{\Phi}_2$ doesn't belong to $L^2(\mathbb{R})$, $\tilde{\Phi}_2$ is a distribution. The scaling coefficients of $\tilde{\Phi}_{2,0}$ is

$$\mathrm{tildeh}_0 = \frac{1+\sqrt{3}}{2}, \quad \tilde{h}_1 = 1, \quad \tilde{h}_2 = \frac{1-\sqrt{3}}{2}.$$

Let x be a real number in $[0,1]$ with the binary expansion

$$x = \sum_{n \in \mathbb{N}} \frac{a_n}{2^n} \quad a_n = 0 \text{ or } 1,$$

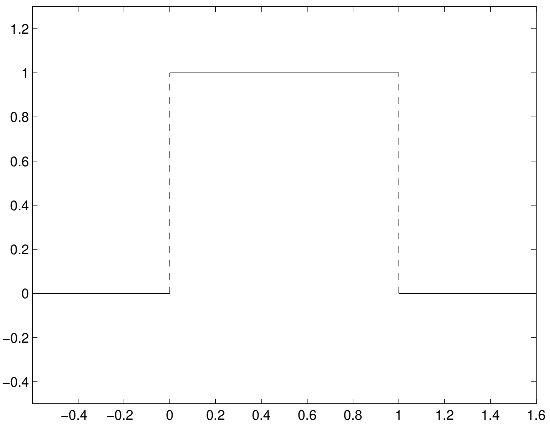

Fig. 1 $\phi_{(1,1)}$

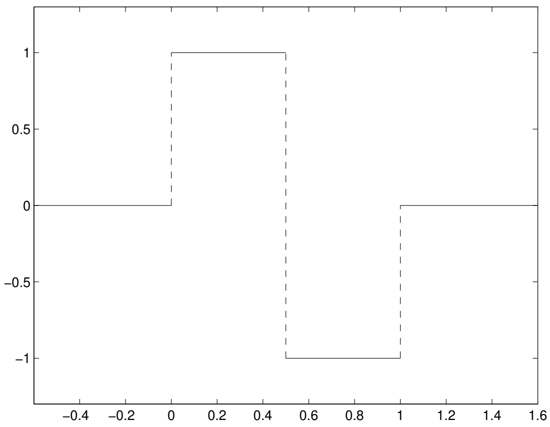

Fig. 2 $\psi_{(1,1)}$

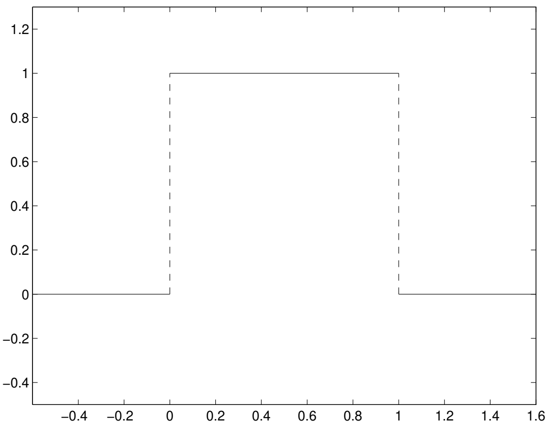

Fig. 3 $\phi_{(1,3)}$

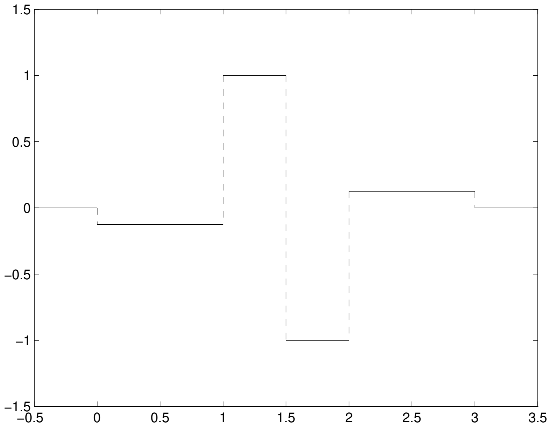

Fig. 4 $\psi_{(1,3)}$

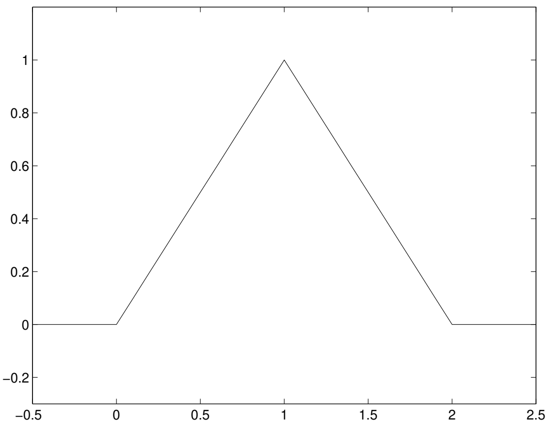

Fig. 5 $\phi_{(2,2)}$

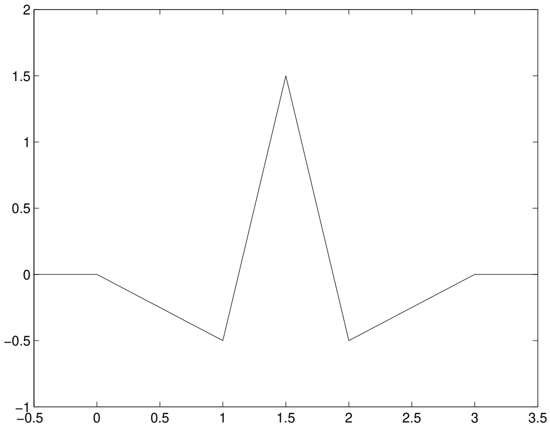

Fig. 6 $\psi_{(2,2)}$

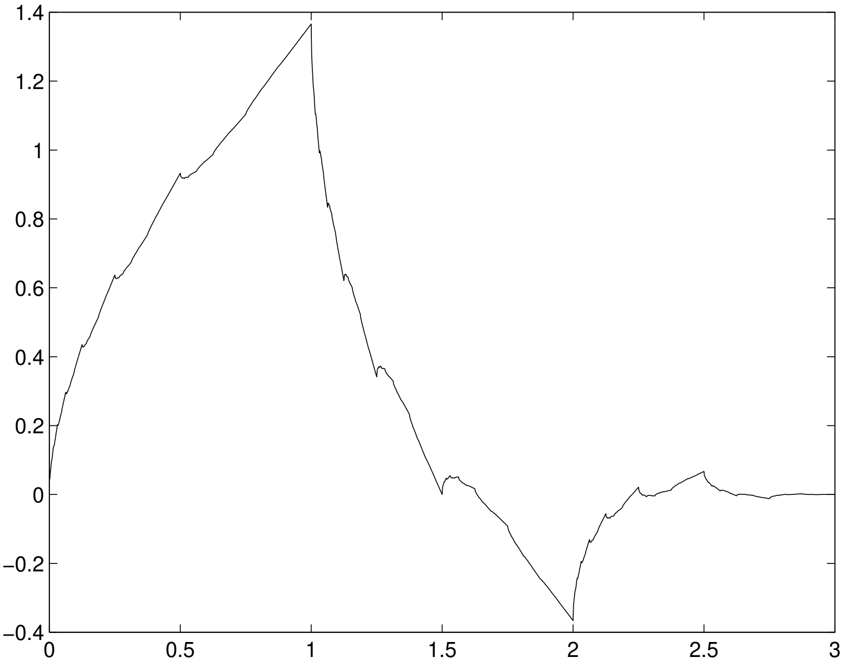

Fig. 7 $\phi_2(x)$

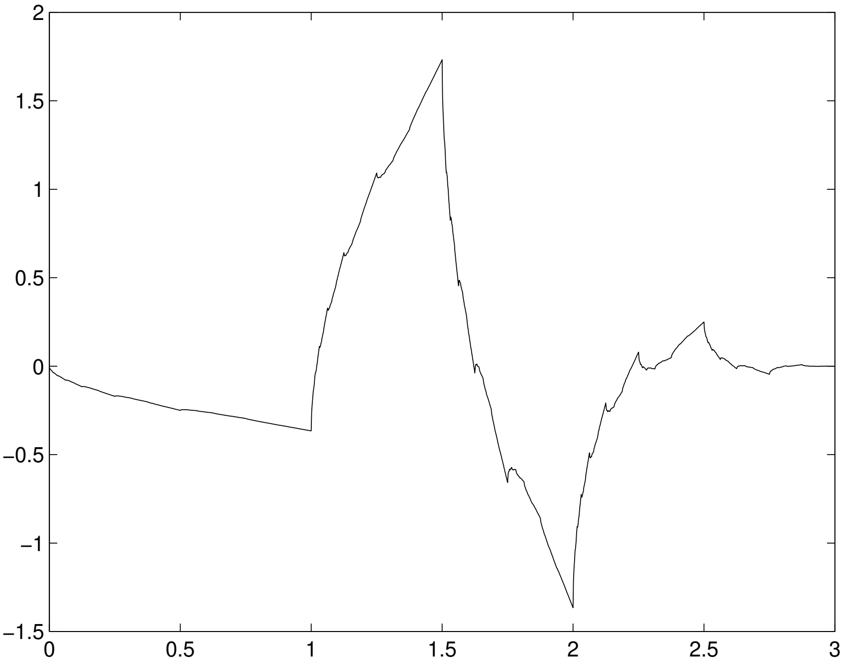

Fig. 8 $\psi_2(x)$

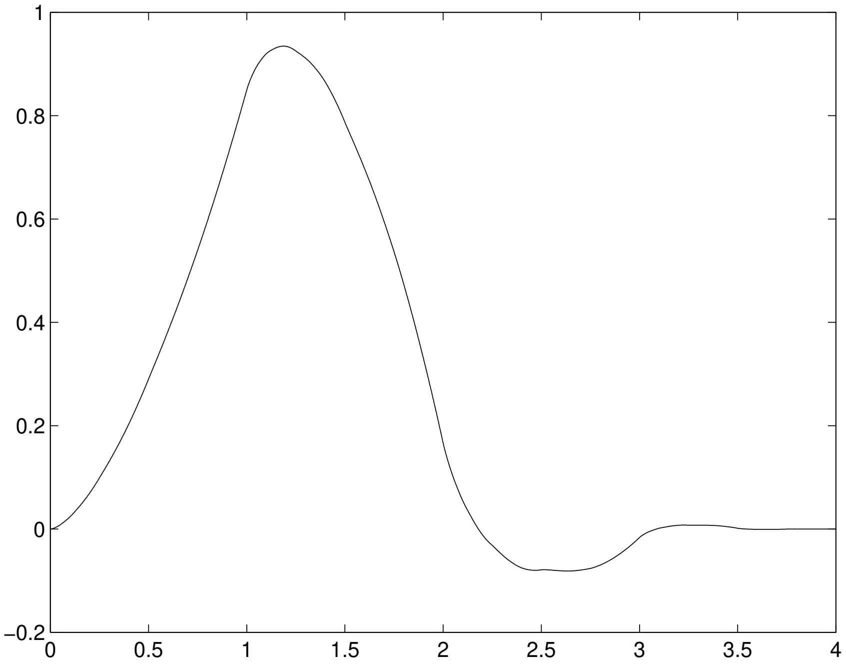

Fig. 9 $\Phi_2(x)$

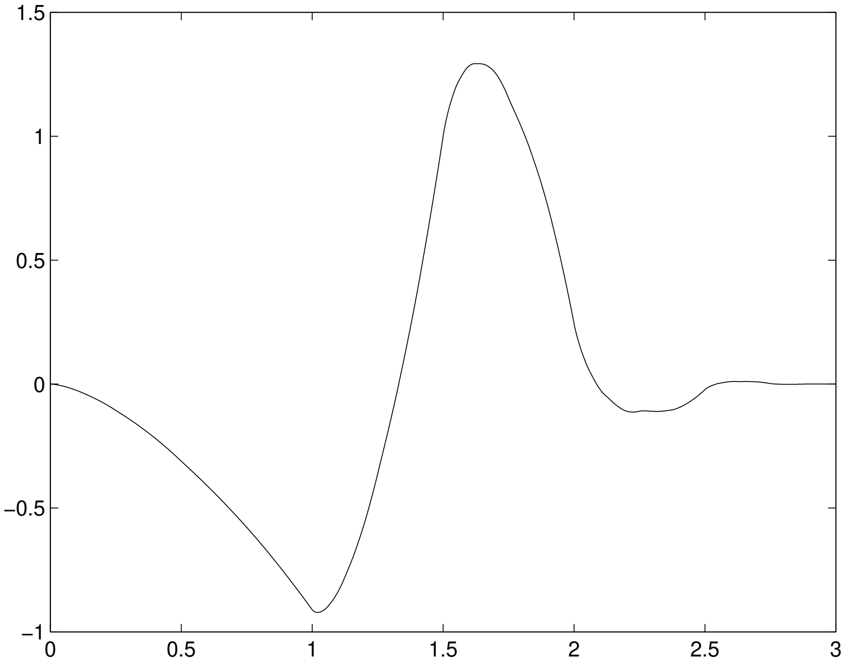

Fig. 10 $\Psi_2(x)$

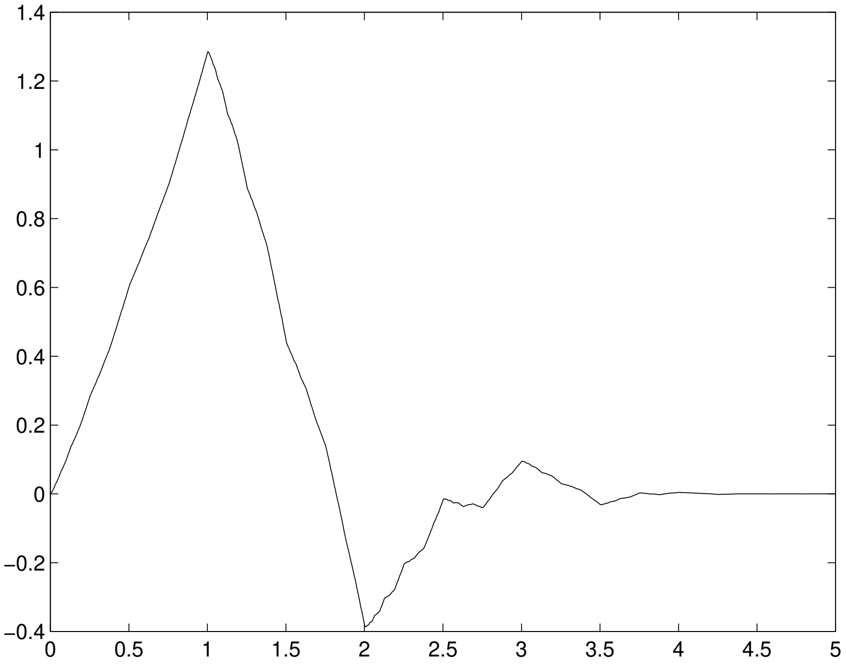

Fig. 11 $\phi_3(x)$

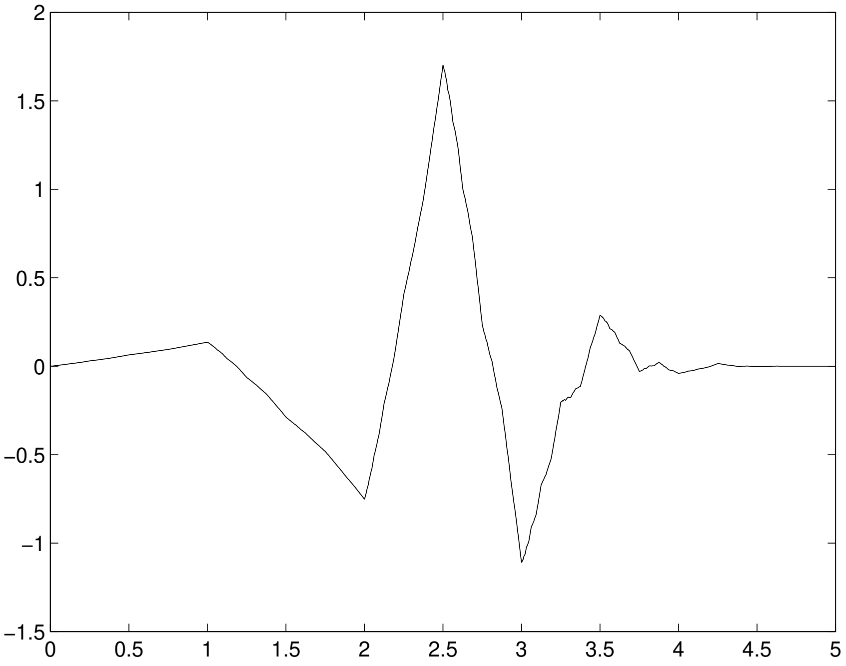

Fig. 12 $\psi_3(x)$

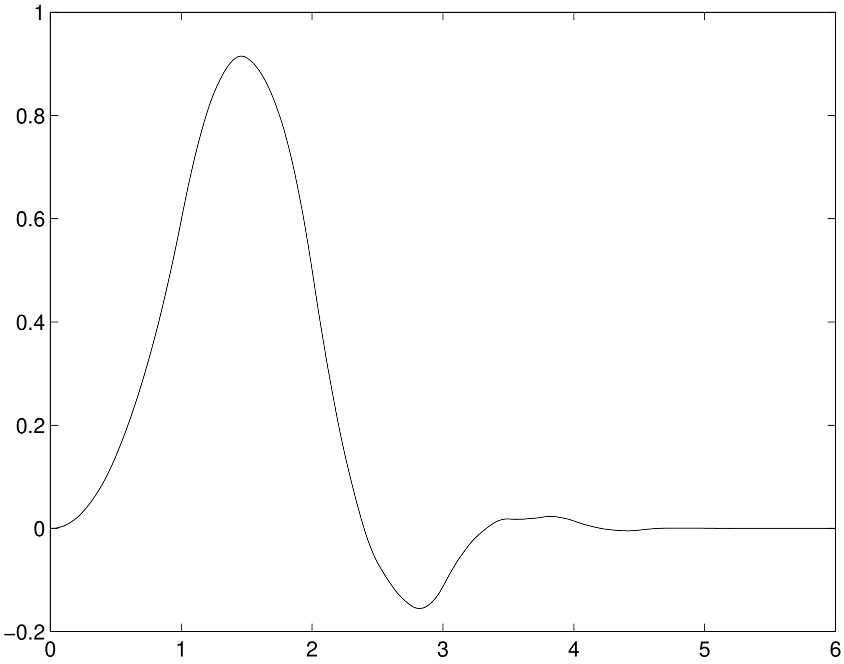

Fig. 13 $\Phi_3(x)$

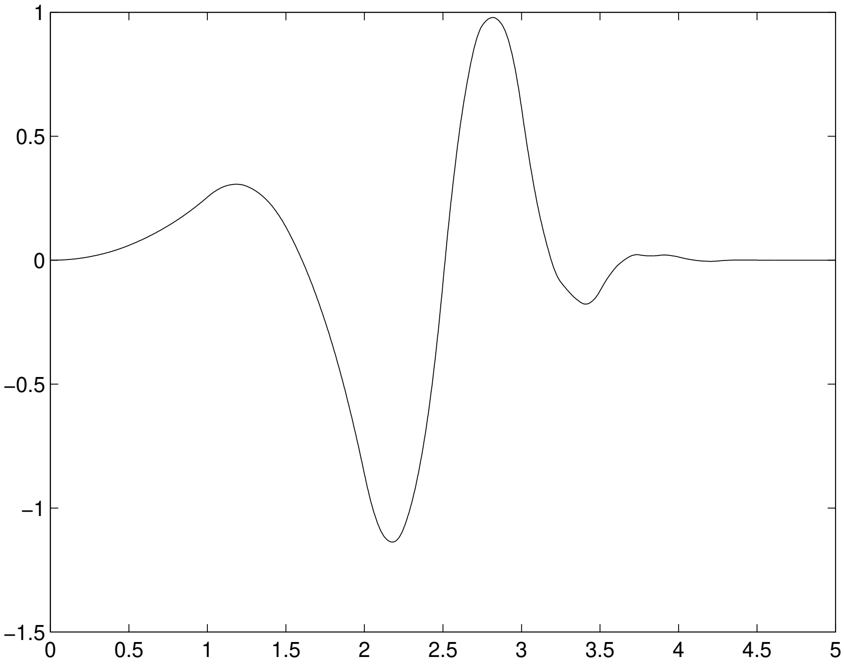

Fig. 14 $\Psi_3(x)$

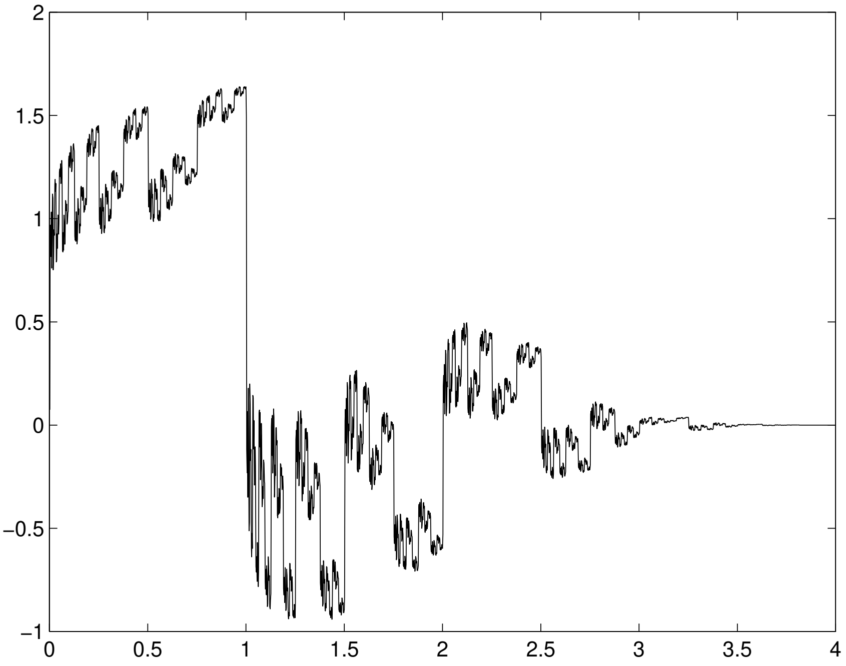

Fig. 15 $\tilde{\Phi}_3(x)$

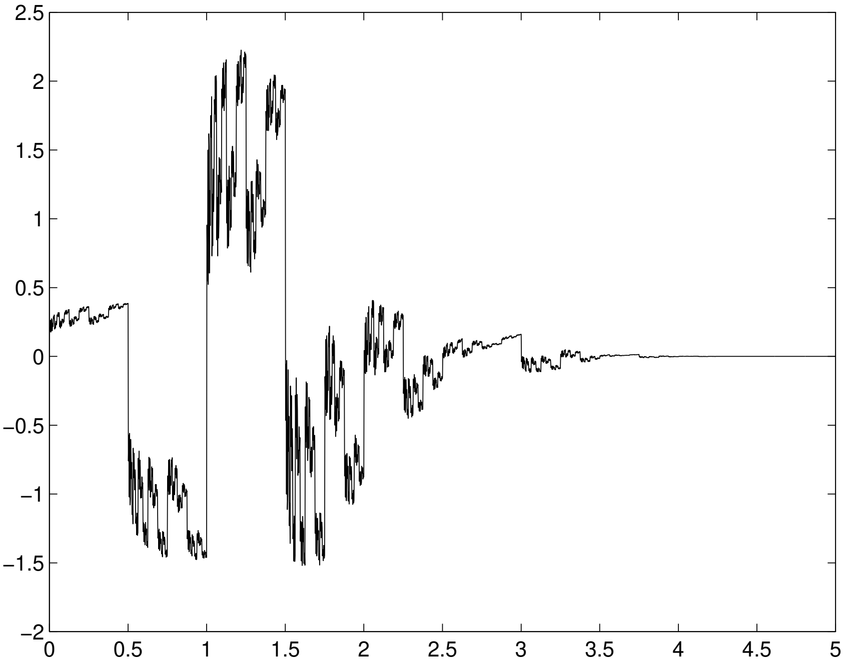

Fig. 16 $\tilde{\Psi}_3(x)$

and τ_n is defined by

$$\tau_n = \sum_{\ell=1}^{n} a_\ell - 1.$$

Let $\alpha = \frac{1+\sqrt{3}}{2}$ and $\beta = \frac{1-\sqrt{3}}{2}$. The explicit form of $\tilde{\Phi}_2$ is given by

$$\tilde{\Phi}_{2,0}(x) = \begin{cases} \sum_{n\in\mathbb{N}} a_n \alpha^{n-\tau_n} \beta^{\tau_n} & \text{for } x \in (0,1] \\ 1 - \tilde{\Phi}_{2,0}(x-1) & \text{for } x \in (1,2] \\ 0 & \text{otherwise.} \end{cases}$$

For case of $p = 3$, ϕ_3 is the orthonormal scaling function and ψ_3 is the orthonormal wavelet. See Figure 11 and 12 respectively for ϕ_3 and ψ_3 in $[0,5]$. We elevate the set $\{\phi_3, \psi_3\}$ to the biorthogornal system $\{\Phi_3, \Psi_3, \tilde{\Phi}_3, \tilde{\Psi}_3\}$. The graphics Φ_3, Ψ_3, $\tilde{\Phi}_3$ and $\tilde{\Psi}_3$ are in Figure 13, 14, 15 and 16, respectively.

5 Application

Section 1 The Galerkin method

Consider the second-order boundary-value problem of Dirichlet type:

$$\begin{cases} -u''(x) = f(x), & \text{for } 0 < x < 1 \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{5.1}$$

If $u(x)$ and $v(x)$ are sufficiently smooth functions, then we have the standard integration-by-parts:

$$\begin{aligned} \int_0^1 -u''(x)v(x)\,dx &= \int_0^1 -v(x)\,du'(x) \\ &= \int_0^1 u'(x)v'(x)\,dx - u'(x)v(x)\Big|_0^1. \end{aligned}$$

Weakly, if $u(x)$ and $v(x)$ are in the Sobolev space $H^1(0,1)$, the integration-by-parts still holds. The space $H^1(0,1)$ is a Hilbert space consists of functions $u(x) \in L^2(0,1)$ whose weak derivatives $u'(x) \in L^2(0,1)$. Let

$$H_0^1(0,1) = \{u \in H^1(0,1) : u(0) = u(1) = 0\}$$

and

$$H_p^1(0,1) = \{u \in H^1(0,1) : u(x) \text{ is periodic}\}.$$

The weak formulation of (5.1) is to find $u \in H_0^1(0,1)$ such that

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v \in H_0^1(0,1). \tag{5.2}$$

We say that $u(x)$ is a weak solution of (5.1) and $v(x)$ is the admissible test function, and $H_0^1(0,1)$ is the class of admissible functions for (5.2). For more details, we refer to the readers the book of Brenner and Scott [1].

Note that the space $H_0^1(0,1)$ is linear and infinite-dimensional. The basis of $H_0^1(0,1)$ is countable, and let $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ be a basis of $H_0^1(0,1)$. If $u \in H_0^1(0,1)$, u can be represented in the form

$$u(x) = \sum_{i=0}^{\infty} u_i \phi_i(x), \tag{5.3}$$

with $(u_i) \in \ell^2$. If we take only a finite number N of terms in the series (5.3), then we will obtain only an approximation u_h of u. The basis function $\{\phi_1, \phi_2, \ldots, \phi_N\}$ define an N-dimensional subset space $V_h \subset H_0^1(0,1)$.

We want to find the numerical approximation $u^h \in H_0^{(N)}(0,1)$ to the weak solution $u \in H_0^1(0,1)$ of the problem (5.1). Thus, instead of taking $v \in H_0^1(0,1)$ in (5.2), we take $v_h \in V_h$, and solve $u_h \in V_h$ for

$$\int_0^1 u_h'(x) v_h'(x)\, dx = \int_0^1 f(x) v_h(x)\, dx \quad \forall v_h \in V_h. \tag{5.4}$$

Let u_h be the solution of (5.4), it is of the form

$$u_a(x) = \sum_{i=1}^{N} u_i \phi_i(x). \tag{5.5}$$

Substitute the basis functions $\{\phi_1, \phi_2, \ldots, \phi_N\}$ into v_h, we have

$$\begin{aligned}\int_0^1 u_a'(x)\phi_j'(x)\, dx &= \int_0^1 \left(\sum_{i=1}^{N} u_i \phi_i'(x)\right) \phi_j'(x)\, dx \\ &= \sum_{i=1}^{N} u_i \left(\int_0^1 \phi_i'(x)\phi_j'(x)\, dx\right).\end{aligned}$$

Then $u_h(x)$ can be determined by the following linear system of equtions.

$$\sum_{i=1}^{N} u_i \left(\int_0^1 \phi_i'(x)\phi_j'(x)\, dx\right) = \int_0^1 f(x)\phi_j(x)\, dx,$$

for $j = 1, \ldots, N$. Hence, to solve the boundary-value problem (5.4) become to solve the linear system

$$Ku = f,$$

where

$$K_{i,j} = \int_0^1 \phi_i'(x)\phi_j'(x)\,dx$$

and

$$f_i = \int_0^1 f(x)\phi_i(x)\,dx.$$

The $N \times N$ symmetric matrix K is called the *stiffness matrix* for problem (5.4) for the basis function ϕ_i. The vector f is called the *load vector*.

The forementioned scheme is called the Galerkin's method. For instance, we can derive finite element method, least square method, and collocation method from the framework of Galerkin's method. In the next section, we will demonstrate a Galerkin-wavelets method for the elementary model problem (5.1).

Section 2 Using $\Phi(x - n)$ to be the basis function

Let $\phi(x)$ be the Daubechies orthonormal scaling function with degree $p = 2$ and $\Phi(x)$ be the elevation of $\phi(x)$. We will solve (5.1) with a little modification: that is, we will assume the periodic boundary condition, and the the admissible space for weak solutions is $H_p^1(0,1)$. We will follow the framework of the Galerkin-wavelets method in [10], the difference is that we will use $\Phi(x)$ as the basis functions. We noticed that an earlier work of Jawerth and Sweldens [6] also contains a particular case of our basis. But they came from a totally different approach and went on to a totally different way of application.

Let $\Phi_{j,k}(x) = \sqrt{2^j}\Phi(2^j x - k)$. We consider the periodic basis functions

$$\{\Phi_{j,1}^*, \Phi_{j,2}^*, \ldots, \Phi_{j,2^j}^*\}$$

for some $j \geq 3$, that is

$$\Phi_{j,n}^*(x) = \Phi_{j,x-n+1} \quad \text{for } 1 \leq n \leq 2^j - 3,$$

$$\Phi^*_{j,2^j-2}(x) = \begin{cases} \Phi_{j,2^j-2}(x) & \text{if } x \in [1-3\cdot 2^{-j}, 1] \\ \Phi_{j,-3}(x) & \text{if } x \in [0, 2^{-j}] \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi^*_{j,2^j-1}(x) = \begin{cases} \Phi_{j,2^j-1}(x) & \text{if } x \in [1-2\cdot 2^{-j}, 1] \\ \Phi_{j,-2}(x) & \text{if } x \in [0, 2\cdot 2^{-j}] \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Phi^*_{j,2^j}(x) = \begin{cases} \Phi_{j,2^j}(x) & \text{if } x \in [1-2^{-j}, 1] \\ \Phi_{j,-1}(x) & \text{if } x \in [0, 3\cdot 2^{-j}] \\ 0 & \text{otherwise.} \end{cases}$$

We define the approximation subset $V_j(0,1)$ by

$$V_j(0,1) = \text{span}\{\Phi^*_{j,1}, \Phi^*_{j,2}, \dots, \Phi^*_{j,2^j}\}.$$

Since $\Phi(x)$ belongs to $\mathcal{C}^1$ and $\Phi(x)$ has support $[0,4]$, $V_j(0,1)$ is a finite-dimensional subset in $H^1_0(0,1)$ for all $j \geq 3$ (we provide the number of basis larger than the number of the nonzero filter). And by two-scaling relation (1.1), we have

$$V_j(0,1) \subset V_{j+1}(0,1) \quad \text{for all } j \geq 3.$$

Moreover (see [10]),

$$\overline{\cup_{j=3}^{\infty} V_j(0,1)} = H^1_p(0,1).$$

We would define the complement $W_j(0,1)$ of $V_j(0,1)$ in $V_{j+1}(0,1)$. And we have

$$\overline{\oplus_{j=3}^{\infty} W_j(0,1)} = H^1_p(0,1).$$

Now, we use Galerkin's method to find an approximate solution u_h of (5.4). We replace V_h by $V_j(0,1)$. Since the basis functions $\Phi^*_k(x)$ are compactly supported, the stiffness matrix $K^{(j)}$ is a band matrix. However, $K^{(j)}$ for the periodic boundary condition is singular. Recall the boundary-value problem (5.1), if u is a solution of (5.1) with periodic boundary condition, then $u + c$ is also a solution of (5.1). There should be another auxiliary condition. We normalize the solution u by $\int_0^1 u(x)\,dx = 1$. Then we have

$$\int_0^1 \left(\sum_{k=1}^{2^j} u_k \Phi^*_{j,k}(x) \right) dx = 1.$$

Since $\int \Phi(x)\,dx = 1$, we have

$$\sum_{k=1}^{2^j} u_k \frac{1}{\sqrt{2^j}} = 1.$$

So we can replace the last row by $K^{(j)}_{2^j,k} = 1$, for all $k = 1, \dots, 2^j$. Correspondingly we must replace the last entry of $\mathbf{f}$ by $\sqrt{2^j}$.

We compute the condition number of the modified matrix $K^{(j)}$, and we will see that $K^{(j)}$ is ill-conditioned. The ill-condition of the stiffness matrix is well known. But when we use $\{\Phi^*_{j,1}, \Phi^*_{j,2}, \dots, \Phi^*_{j,2^j}\}$ to be the basis function, we can use the biorthogonal structure to transform the matrix $K^{(j)}$. That is, we change the basis from

$$\{\Phi^*_{j,1}, \Phi^*_{j,2}, \dots, \Phi^*_{j,2^j}\} \tag{5.6}$$

to

$$\{\Phi^*_{j-1,1}, \Phi^*_{j-1,2}, \dots, \Phi^*_{j-1,2(j-1)} \Psi^*_{j-1,1}, \Psi^*_{j-1,2}, \dots, \Psi^*_{j-1,2(j-1)}\}, \tag{5.7}$$

where $\Psi^*_{j,k}$ are the periodized wavelets defined similar as $\Phi^*_{j,k}$.

Constructively, when we change the basis from (5.6) to (5.7), we just multiply a *decomposition matrix* C and C^t on both sides of $K^{(j)}$. The decomposition matrix C is of the form (for case $j = 3$)

$$\begin{pmatrix}
h_0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 \\
0 & 0 & h_0 & h_1 & h_2 & h_3 & h_4 \\
h_3 & h_4 & 0 & 0 & h_0 & h_1 & h_2 \\
h_1 & h_2 & h_3 & h_4 & 0 & 0 & h_0 \\
g_0 & g_1 & g_2 & 0 & 0 & 0 & 0 \\
0 & 0 & g_0 & g_1 & g_2 & 0 & 0 \\
0 & 0 & 0 & 0 & g_0 & g_1 & g_2 \\
g_2 & 0 & 0 & 0 & 0 & g_0 & g_1
\end{pmatrix},$$

where h_n are the scaling coefficients of $\Phi(x)$ and g_n are the wavelets coefficients of $\Psi(x)$. Denote the original stiffness matrix $K^{(j)}$ as $K^{(j,0)}$, and $K^{(j,1)}$ is that associated to the basis (5.7). Repeating the change of basis from (5.6) to (5.7), we get $K^{(j,2)}$, $K^{(j,3)}$, and so on. The following table shows us the original condition number of $K^{(j,0)}$ for

$j = 6, \dots, 10$, and the condition number of $K^{(j,1)}$ to $K^{(j,7)}$ (note that we need $j - 7 \geq 3$).

[m*n]	$64 * 64$	$128 * 128$	$256 * 256$	$512 * 512$	$1024 * 1024$
0	1466.07	8287.14	46870.03	265125.79	1499760.24
1	735.25	2932.16	11719.78	46870.27	265125.78
2	186.01	735.25	2932.16	11719.78	46870.27
3	48.68	186.01	735.25	2932.16	11719.78
4		48.68	186.01	735.25	2932.16
5			48.68	186.01	735.25
6				48.68	186.01
7					48.68

We see that when we change $K^{(j,0)}$ to $K^{(j,1)}$, the condition number becomes smaller (divided by 2). Because $\Psi'(x)$ are orthogornal, and

$$\langle \Phi'(x), \Psi'(x-n) \rangle = 0 \quad \forall n \in \mathbb{Z},$$

the right-low quarter of $K^{(j,1)}$ is a diagonal matrix, and both of the right-top and left-low quarters of $K^{(j,1)}$ are zero. That is why the condition number of $K^{(j,1)}$ is better then $K^{(j,0)}$. Actually the condition number of $K^{(j,1)}$ is determined by the left-top quarter, which is exactly the $K^{(j-1,0)}$. Moreover, the condition number of $K^{(j_1,\ell_1)}$ is the same as that of $K^{(j_2,\ell_2)}$, if $j_1 - \ell_1 = j_2 - \ell_2 \leq 3$. Then we don't worry about the increasing of the condition number, when the size of the stiffness matrix increases.

Index

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